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Tensor Integrals

By William Fabian

(*Received* 13*th May*, 1955.)

1. *Introduction.*

A form of integration of tensors will be introduced here, which will preserve the character of a tensor when so integrated.

Let $T^{u_1 \cdots u_p}_{v_1 \cdots v_q}$ be the components of a given tensor, defined in the co-ordinate system x^r at all points of a curve l, in a space V_n of affine connection [1] with connection coefficients L^i_{jk}. We consider the n^{p+q} differential equations

$$T^{u_1 \cdots u_p}_{v_1 \cdots v_q} = \frac{\delta}{\delta t}\left(X^{u_1 \cdots u_p}_{v_1 \cdots v_q}\right),$$

where the quantities on the right are the intrinsic derivatives [2] of the components $X^{u_1 \cdots u_p}_{v_1 \cdots v_q}$ of a tensor with respect to a parameter t which defines l.

Definition. If these differential equations have a solution

$$X^{u_1 \cdots u_p}_{v_1 \cdots v_q} = Z^{u_1 \cdots u_p}_{v_1 \cdots v_q}$$

(the indices taking the values 1 *to* n*), then* $Z^{u_1 \cdots u_p}_{v_1 \cdots v_q}$ *will be called a tensor integral of* $T^{u_1 \cdots u_p}_{v_1 \cdots v_q}$ *along* l *with respect to* t*, and will be written*

$$\int T^{u_1 \cdots u_p}_{v_1 \cdots v_q}\, \delta t.$$

2. *Properties of tensor integrals.*

The quantities $Z^{u_1 \cdots u_p}_{v_1 \cdots v_q}$ transform as components of a tensor and

[1] Eisenhart, *Non-Riemannian Geometry* (1927), chapter I.

[2] These intrinsic derivatives must not be confused with ordinary differential coefficients. For definitions see Eisenhart, *op. cit.*, chapter I. All quantities used in this paper are real.

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represent therefore a tensor which has the same transformation coefficients as the tensor $T^{u_1 \cdots u_p}_{v_1 \cdots v_q}$.

The tensor integral $\int T^{u_1 \cdots u_p}_{v_1 \cdots v_q} \delta t$, *if it exists, represents therefore the components of a tensor of the same type and order as the tensor* $T^{u_1 \cdots u_p}_{v_1 \cdots v_q}$.

From theorems on differential equations[1] we can deduce immediately the following two theorems:—

THEOREM 1. *If the quantities* dx^r/dt, *the coefficients of the connection of the space and the components* $T^{u_1 \cdots u_p}_{v_1 \cdots v_q}$ *of a tensor in the co-ordinate system* x^r *are continuous functions of the parameter* t *which defines a curve* l, *then a tensor integral* $\int T^{u_1 \cdots u_p}_{v_1 \cdots v_q} \delta t$ *exists, representing components of a tensor which are continuous along* l.

THEOREM 2. *If the co-ordinates* x^r, *the coefficients of the connection of the space and the components* $T^{u_1 \cdots u_p}_{v_1 \cdots v_q}$ *of a tensor in this co-ordinate system are analytic functions of the parameter* t *which defines a curve* l, *then a tensor integral* $\int T^{u_1 \cdots u_p}_{v_1 \cdots v_q} \delta t$ *exists as a set of analytic functions of* t *along* l.

3. *Types of tensor integrals.*

Definition. Quantities $P^{u_1 \cdots u_p}_{v_1 \cdots v_q}$ satisfying the equations

$$\frac{\delta}{\delta t}\left(X^{u_1 \cdots u_p}_{v_1 \cdots v_q}\right) = 0 \tag{1}$$

will be said to be the components of tensors *parallel with respect to* l *such that any one of these tensors may be obtained from any other by parallel displacement along this curve.*

Hence, if $X^{u_1 \cdots u_p}_{v \cdots v_q} = A^{u_1 \cdots u_p}_{v_1 \cdots v_q}$ be a *particular* solution of the equations

$$T^{u_1 \cdots u_p}_{v_1 \cdots v_q} = \frac{\delta}{\delta t}\left(X^{u1 \cdots u_p}_{v_1 \cdots v_q}\right), \tag{2}$$

[1] Goursat, *Mathematical Analysis*, translated by Hedrick, vol. II (1916).

then, by (1), the *complete* solution of equations (2) is given by

$$X^{u_1 \cdots u_p}_{v_1 \cdots v_q} = A^{u_1 \cdots u_p}_{v_1 \cdots v_q} - P^{u_1 \cdots u_p}_{v_1 \cdots v_q}.$$

(a) Thus we obtain the theorem:—

THEOREM 3. *Under the conditions of either Theorem* 1 *or Theorem* 2, *all tensor integrals of* $T^{u_1 \cdots u_p}_{v_1 \cdots v_q}$ *with respect to* t *are given by*

$$\int T^{u_1 \cdots u_p}_{v_1 \cdots v_q} \delta t = A^{u_1 \cdots u_p}_{v_1 \cdots v_q} - P^{u_1 \cdots u_p}_{v_1 \cdots v_q},$$

where $A^{u_1 \cdots u_p}_{v_1 \cdots v_q}$ *form a particular tensor integral of the given tensor with respect to* t, *and* $P^{u_1 \cdots u_p}_{v_1 \cdots v_q}$ *are the components of tensors obtained by parallel displacement along* l *of an arbitrary initial tensor.*

(b) Under the conditions of Theorem 3, we define the tensor integral of $T^{u_1 \cdots u_p}_{v_1 \cdots v_q}$ along l between the limits t_0 and t_1 by

$$\int_{t_0}^{t_1} T^{u_1 \cdots u_p}_{v_1 \cdots v_q} \delta t = \left(A^{u_1 \cdots u_p}_{v_1 \cdots v_q}\right)_{t_1} - \left(P^{u_1 \cdots u_p}_{v_1 \cdots v_q}\right)_{t_1},$$

where $\left(A^{u_1 \cdots u_p}_{v_1 \cdots v_q}\right)_{t_1}$ are the components of the tensor $A^{u_1 \cdots u_p}_{v_1 \cdots v_q}$ at a point t_1 on l, and $\left(P^{u_1 \cdots u_p}_{v_1 \cdots v_q}\right)_{t_1}$ are the components of a tensor at the same point t_1 obtained by parallel displacement of the tensor $\left(A^{u_1 \cdots u_p}_{v_1 \cdots v_q}\right)_{t_0}$ along l from the point t_0 to t_1.

(c) The tensor integral of a tensor of order zero is the ordinary integral of the function representing this tensor.

4. *Tensor integration by parts.*

From the definition and properties of tensor integrals (stated in sections 1 and 2) it follows at once that *tensor integration with respect to a parameter defining a curve and intrinsic differentiation with respect to this parameter are inverse operations.*

THEOREM 4. *Let the quantities dx^r/dt and the coefficients of the connection of the space be continuous along a curve l defined by the parameter t.*

Let the components U of a tensor in the co-ordinate system x^r have continuous intrinsic derivatives $\delta U/\delta t$ along l, and let the components W of another tensor in this co-ordinate system be continuous along l.

Then

$$\int_{t_0}^{t_1} UW\,\delta t = \left(U\int W\delta t\right)_{t_1} - \left(P\right)_{t_1} - \int_{t_0}^{t_1}\left[\frac{\delta U}{\delta t}\int W\delta t\right]\delta t,$$

where $(P)_{t_1}$ are the components of a tensor at the point t_1 obtained by parallel displacement of the tensor $\left(U\int W\delta t\right)_{t_0}$ along l from the point t_0 to t_1.

Proof. Put $\int W\delta t = M$.

We have

$$\frac{\delta}{\delta t}\left(UM\right) = U\frac{\delta M}{\delta t} + M\frac{\delta U}{\delta t}$$

Hence

$$\int_{t_0}^{t_1}\frac{\delta}{\delta t}\left(UM\right)\delta t = \int_{t_0}^{t_1} U\frac{\delta M}{\delta t}\,\delta t + \int_{t_0}^{t_1} M\frac{\delta U}{\delta t}\,\delta t,$$

that is,

$$\left(UM\right)_{t_1} - \left(P\right)_{t_1} = \int_{t_0}^{t_1} U\frac{\delta M}{\delta t}\delta t + \int_{t_0}^{t_1} M\frac{\delta U}{\delta t}\,\delta t.$$

The conclusion follows.

5. *The m^{th} tensor integral of a tensor.*

THEOREM 5. *Let the quantities dx^r/dt and the coefficients of the connection of the space be continuous along a curve l defined by the parameter t.*

If the components W of a tensor in the co-ordinate system x^r are continuous along l, then, for all positive integers m, the m^{th} tensor integral of W with respect to t between t_0 and z is given by

$$\frac{1}{(m-1)!}\int_{t_0}^{z}(z-t)^{m-1}\,W\delta t.$$

Proof. $(z-t)^{m-1}$ is a tensor of order zero. Applying Theorem 4, we obtain

$$\frac{1}{(m-1)!}\int_0^z (z-1)^{m-1}\, W\delta t$$

$$= \left(\frac{(z-t)^{m-1}}{(m-1)!}\int_0^t W\delta t\right)_z - (P)_z + \int_0^z \left[\frac{(z-t)^{m-2}}{(m-2)!}\int_0^t W\delta t\right]\delta t$$

$$= \frac{1}{(m-2)!}\int_{t_0}^z \left[(z-t)^{m-2}\int_{t_0}^t W\delta t\right]\delta t,$$

since the components $(P)_z$ belong to a tensor obtained by parallel displacement of a zero tensor and are therefore zero for all values of the indices.

Continuing this process of tensor integration by parts, we obtain the conclusion.

6. *Fractional tensor integrals and intrinsic derivatives.*[1]

Now let m have any real value, but let the conditions of Theorem 5 be otherwise satisfied. Then we define the m^{th} tensor integral, or the $(-m)^{\text{th}}$ intrinsic derivative, of W with respect to z by [2]

$$\left(\frac{\delta}{\delta z}\right)^{-m} W = \frac{1}{\Gamma(m+c)}\left(\frac{\delta}{\delta z}\right)^c \int_0^z (z-t)^{m+c-1}\, W\delta t,$$

where c is the least integer greater than or equal to zero such that $m + c \geq 1$.

Since tensor integration and intrinsic differentiation do not alter the type and order of a tensor, and $(z-t)^{m+c-1}$ is a tensor of order zero, it follows that, *for any real value of* m, *the* m^{th} *tensor integral* $\left(\frac{\delta}{\delta z}\right)^{-m} W$ *is a tensor of the same type and order as the tensor* W.

7. *Tensor expansions.*[3]

THEOREM 6. *Let the quantities* dx^r/dt *and the coefficients of the connection of the space be continuous along a curve* l *defined by the parameter* t. *Let the derivatives* $\delta^m T^{u_1 \ldots u_p}_{v_1 \ldots v_q} \Big/ \delta t^m$ *of the components* $T^{u_1 \ldots u_p}_{v_1 \ldots v_q}$ *of*

[1] See my paper in *Phil. Mag.* (7), XX (1935), 781-789.

[2] The gamma function $\Gamma(m+c)$ is not to be confused with the Christoffel symbols.

[3] These expansions correspond to the Taylor series for ordinary functions.

a tensor in the co-ordinate system x^r be continuous along l for all positive and zero integers m. Then, in the interval of convergence,

$$\left(T^{u_1 \ldots u_p}_{v_1 \ldots v_q}\right)_z = \sum_{m=0}^{\infty} \frac{(z-t_0)^m}{m!} \left({}^{(m)}U^{u_1 \ldots u_p}_{v_1 \ldots v_q}\right)_z,$$

where $\left({}^{(m)}U^{u_1 \ldots u_p}_{v_1 \ldots v_q}\right)_z$ *are the components of a tensor obtained by parallel displacement of the tensor* $\left(\frac{\delta^m T^{u_1 \ldots u_p}_{v_1 \ldots v_q}}{\delta t^m}\right)_{t_0}$ *along l from t_0 to z.*

Proof. Write T for $T^{u_1 \ldots u_p}_{v_1 \ldots v_q}$, and apply the process of tensor integration by parts of Theorem 4 to the tensor integral $\int_{t_0}^{z} 1 T \delta t$. Since 1 is a tensor of order zero (so that its tensor integral is its ordinary integral), we have, by Theorem 4,

$$\int_{t_0}^{z} 1 T \delta t = (-(z-t)\,T)_z - (P)_z + \int_{t_0}^{z} (z-t) \frac{\delta T}{\delta t}\, \delta t$$

$$= -(P)_z + \int_{t_0}^{z} (z-t) \frac{\delta T}{\delta t}\, \delta t, \tag{1}$$

where $(P)_z$ is the tensor at the point z obtained by parallel displacement of the tensor $(-(z-t)\,T)_{t_0}$ along l from t_0 to z.

We have also
$$\frac{\delta P}{\delta t} = 0$$
and
$$\frac{\delta}{\delta t}({}^{(0)}U) = 0, \tag{2}$$

where ${}^{(0)}U$ stands for ${}^{(0)}U^{u_1 \ldots u_p}_{v_1 \ldots v_q}$.

At the point $t = t_0$,

$$P = (-(z-t)\,T)_{t_0} = -(z-t_0)\,({}^{(0)}U)_{t_0}. \tag{3}$$

By (2) and (3), $P = -(z-t_0)\,({}^{(0)}U)$ is a solution of $\delta P/\delta t = 0$.

Hence by (1) we have

$$\int_{t_0}^{z} T \delta t = (z-t_0)\,({}^{(0)}U)_z + \int_{t_0}^{z} (z-t) \frac{\delta T}{\delta t}\, \delta t$$

$$= (z - t_0)\,({}^{(0)}U)_z + \ldots + \frac{(z-t_0)^m}{m!}\,({}^{(m-1)}U)_z + \ldots$$

$$\ldots + \int_{t_0}^{z} \frac{(z-t)^m}{m!}\,\frac{\delta^m T}{\delta t^m}\,\delta t$$

on continuing this tensor integration by parts, where ${}^{(m-1)}U$ stands for ${}^{(m-1)}U^{u_1 \ldots u_p}_{v_1 \ldots v_q}$.

Hence $(T)_z \equiv \dfrac{\delta}{\delta z}\displaystyle\int_{t_0}^{z} T\,\delta t$

$$= \sum_{n=1}^{m} \frac{(z-t_0)^{n-1}}{(n-1)!}\,({}^{(n-1)}U)_z + \frac{\delta}{\delta z}\int_{t_0}^{z} \frac{(z-t)^m}{m!}\,\frac{\delta^m T}{\delta t^m}\,\delta t.$$

This gives the required series, which converges if

$$\frac{\delta}{\delta z}\int_{t_0}^{z} \frac{(z-t)^m}{m!}\,\frac{\delta^m T}{\delta t^m}\,\delta t \to 0$$

when $m \to \infty$.

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A Determinantal Expansion for a Class of Definite Integral

Part 4.

By L. R. Shenton

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1. We shall show in this part the relation of generalised C.F.'s to ordinary C.F.'s, in the main confining our attention to Stieltjes type fractions. Moreover we shall bring out the part played by Parseval's theorem in our development of the subject, and a property of extremal solutions of the Stieltjes moment problem given by M. Riesz.[1]

2. The Stieltjes moment problem[2] (S.M.P.) concerns itself with finding a bounded non-decreasing function $\psi(x)$ in $(0, \infty)$ such that

$$\int_0^\infty x^n d\psi(x) = \mu_n, \; n = 0, 1, 2, \ldots,$$

where the μ's are real. The solution offered by Stieltjes depends upon the characteristics of the C.F. *associated* with the formal expansion

$$F(z, \psi) = \int_0^\infty \frac{d\psi(x)}{z + x} \sim \frac{\mu_0}{z} - \frac{\mu}{z^2} + \frac{\mu_2}{z^3} - \ldots, \tag{1}$$

namely

$$F(z, \psi) \sim \cfrac{1}{az_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 z + \cfrac{1}{a_4 + \cfrac{1}{a_5 z + \cdots}}}}} \tag{2}$$

and the *corresponding* C.F. (obtained by contraction)

$$F(z, \psi) \sim \cfrac{\lambda_1}{z + C_1 - \cfrac{\lambda_2}{z + C_2 - \cfrac{\lambda_3}{z + C_3 - \cdots}}} \tag{3}$$

A necessary and sufficient condition for the existence of a solution of the S.M.P. is that $a_j > 0, \quad j = 1, 2, 3, \ldots$; the solution is unique if $\sum_1^\infty a_j$ diverges. If $\sum_1^\infty a_j$ converges there may be an

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[1] M. Riesz, "Sur le problème des moments," *Arkiv for matematik, astronomi och fysik*, 16 (12), 1-21 ; 16 (19), 1-21 ; 17 (16) 1-52.

[2] See, for example, J. A. Shohat and J. D. Tamarkin, *The Problem of Moments* (American Mathematical Society Surveys No. 1, 1943).

infinity of solutions; of these there are two called extremal solutions, which have the property that $\{(\bar{\omega}_{2s}(-z)\}$ and $\{\bar{\omega}_{2s+1}(-z)\}$ are orthogonal systems with respect to them, where $\bar{\chi}_s(z)/\omega_s(z)$ is the s^{th} convergent of (2).

The moments μ_j and the elements a_j are related as follows:

$$a_{2j} = \frac{\Delta_j^2}{\Delta_{j-1}^1 \Delta_j^1}, \qquad a_{2j+1} = \frac{\Delta_j'^2}{\Delta_j \Delta_{j+1}} \tag{4}$$

where $\begin{cases} \Delta_j = |\mu_0, \mu_2, \cdots \mu_{2j-2}|, \quad \Delta'_j = |\mu_1, \mu_3, \cdots \mu_{2j-1}| \\ \quad > 0. \qquad\qquad\qquad\quad > 0 \\ \Delta_0 = 1, \quad \Delta_1 = \mu_0, \quad \Delta'_0 = 1, \quad \Delta'_1 = \mu_1. \end{cases}$

Stieltjes showed that when there is a solution of the S.M.P. then

$$F(z, \psi) - \bar{\chi}_{2s}(z)/\bar{\omega}_{2s}(z) = \min_{p_s} \int_0^\infty \frac{p_s^2(x) d\psi(x)}{z+x}$$

$$z\{\chi_{2s+1}(z)/\bar{\omega}_{2s+1}(z) - F(z, \psi)\} = \min_{p_s} \int_0^\infty \frac{p_s^2(x) x d\psi(x)}{z+x}$$

where $p_s(-z) = 1$, $z > 0$, and the minimum is taken over all polynomials of degree less than or equal to s. For our present discussion it is important to write these in the form

$$F(z, \psi) - \bar{\chi}_{2s+2}(z)/\bar{\omega}_{2s+2}(z) = \min_{\Pi_s} \int_0^\infty (z+x) d\psi(x) \left\{\frac{1}{z+x} - \Pi_s(x)\right\}^2 \tag{5}$$

$$z\{\bar{\chi}_{2s+3}(z)/\bar{\omega}_{2s+3}(z) - F(z, \psi)\} = \min_{\Pi_s} \int_0^\infty x(z+x) d\psi(x) \left\{\frac{1}{z+x} - \Pi_s(x)\right\}^2, \tag{6}$$

the forms (5) and (6) showing the obvious relation to Parseval's theorem. Indeed from a formal point of view all we have to do to find the even part of (2) is to write down the Parseval expansion for

$$\int_0^\infty f(x)^2 d\bar{\psi}(x) \quad \text{where } f(x) = (z+x)^{-1}, \quad \bar{\psi}(x) = \int_0^x (z+t) d\psi(t)$$

and the Tchebicheffian orthogonal polynomials with respect to $d\psi(x)$ are related to those for $d\bar{\psi}(x)$ by the theorem of Christoffel (Part 1, 14). Similarly the odd part of (2) follows from $\int_0^\infty f(x)^2 d\hat{\psi}(x)$ where $\hat{\psi}(x) = \int_0^x t(z+t) d\psi(t)$.

On the other hand it has been proved by Riesz (*loc. cit.*) that if $\psi(x)$ is an extremal solution of the S.M.P. (or the unique solution) then Parseval's theorem applies to all $f(x) \epsilon L^2_{\psi}$. An extension of this which we require is the following: If $\psi(x)$ is an extremal solution of the S.M.P., then Parseval's theorem applies to $f(x) \epsilon L^2_{\hat{\psi}}$ where $\hat{\psi}(x) = \int_0^x \Pi(t) d\psi(t)$ and $\Pi(t)$ is a non-negative polynomial of fixed degree which is not identically zero. Clearly $\hat{\psi}(x)$ is bounded and non-decreasing and the moments are given by

$$\mu'_s = \int_0^\infty x^s d\hat{\psi}(x) = \int^\infty x^s \Pi(x) d\psi(x)$$
$$= \sum_{(r)} \gamma_r \mu_{r+s} \text{ where } \Pi(x) = \sum_{(r)} \gamma_r x^r.$$

Thus if $\psi(x)$ is the unique solution for a given sequence $\{\mu_s\}$ (assuming such a solution exists) then $\int_0^x \Pi(t) d\psi(t)$ is the unique solution for the sequence $\{\Pi(\mu)\}$, provided $\Pi(x)$ is a non-negative polynomial. It is of interest to recall that the S.M.P. is determined in the particular cases [1]

$$\mu_s = \int_0^\infty x^s x^{b-1} \exp(-kx^a)\, dx,\ b > 0, \quad k > 0,\ a \geqq \tfrac{1}{2} \tag{7}$$

$$\mu_s = \int_0^\infty x^s f(x)\, x^{b-1} \exp(-kx^a) dx \tag{8}$$

where $f(x)$ is a positive bounded function on (a, ∞), $a > 0$. Moreover quite apart from the theory of continued fractions, Hardy [2] proved that the S.M.P. is determined for $\psi(t) = \int_0^t \phi(x) dx$, $\phi(x) \geqq 0$, provided $\int_0^\infty [\phi(t)]^q e^{\delta \sqrt{t}} dt < \infty$ for $q \geqq 1$, $\delta > 0$.

It will be noticed that the uniqueness of the moment problem $\mu_n = \int_0^\infty x^n \phi(x) dx$ depends upon the order of magnitude of $\phi(x)$ for large positive x. Thus $\phi(x) = \exp - x^{\frac{1}{4}}$ does not approach zero rapidly enough for $x \to \infty$, and the Stieltjes C.F. corresponding to $\int_0^\infty \frac{\exp - x^{\frac{1}{4}} dx}{z + x} = F(z)$ diverges by oscillation. But making the substitu-

[1] T. J. Stieltjes, *Oeuvres Complètes*, Vol. 2, pp. 505-506, 518-520.

[2] G. H. Hardy, "On Stieltjes' 'problème des moments,'" *Messenger of Mathematics*, 46 (1917), 175-182; 47 (1917), 81-88.

tion $x = t^2$ we have $F(z) = \int_0^\infty \frac{e^{-\sqrt{t}}dt^2}{z+t^2}$ and we shall show in the sequel that the second order C.F. for $F(z)$ converges. In general it may be remarked that if $\psi(x)$ is the solution of a determined S.M.P. then the C.F. (2), with elements given by (4) in terms of the moments, converges for $z > 0$; but the Stieltjes C.F. corresponding to $\int_0^\infty \frac{d\psi(x)}{z+x^s} = F(z)$ may not converge, s being a positive integer greater than unity. However, the s^{th} order C.F. corresponding to $F(z)$ does converge in this case.

2.0 We now state some properties of the convergents of Stieltjes C.F.'s which we require. We consider the expansions

$$F(z) = \int_0^\infty \frac{d\psi(x)}{x+z} = \frac{b_1}{z} + \frac{b_2}{1} + \frac{b_3}{z} + \frac{b_4}{1} + \cdots \quad \left(= \frac{\chi_s(z)}{\omega_s(z)} \text{ as } s \to \infty\right) \tag{9}$$

$$= \frac{b_1}{z+b_2} - \frac{b_2 b_3}{z+b_3+b_4} - \frac{b_4 b_5}{z+b_5+b_6} - \cdots \quad \left(= \frac{\chi_{2s}(z)}{\omega_{2s}(z)} \text{ as } s \to \infty\right) \tag{10}$$

assumed to be convergent for $z > 0$. Then we have the recurrence relations

$$\left.\begin{aligned} \omega_{2s}(z) &= (z + b_{2s-1} + b_{2s})\,\omega_{2s-2}(z) - b_{2s-1}b_{2s-2}\omega_{2s-4}(z) \\ \omega_{2s+1}(z) &= (z+b_{2s}+b_{2s+1})\omega_{2s-1}(z) - b_{2s}\,b_{2s-1}\omega_{2s-3}(z) \end{aligned}\right\} \quad s = 2, 3, \ldots \tag{11}$$

and similarly for $\chi_s(z)$ with $\chi_0 = 0$, $\chi_1 = b_1$; $\omega_0 = 1$, $\omega_1 = z$, from which we derive the determinantal relations [1]

---

[1] The following abbreviated notation for alternant types of determinants will be used throughout:

$$\left|\, A_r(z_1),\quad B_s(z_2),\quad C_t(z_3) \,\right|^{+} = \begin{vmatrix} A_r(z_1) & B_r(z_2) & C_r(z_3) \\ A_s(z_1) & B_s(z_2) & C_s(z_3) \\ A_t(z_1) & B_t(z_2) & C_t(z_3) \end{vmatrix}$$

where any functional symbol cannot be separated from its argument. Thus

$$\left|\, \chi_{2r}(z_1),\ \omega_{2r+2}(z_2) \,\right|^{+} = \begin{vmatrix} \chi_{2r}(z_1) & \omega_{2r}(z_2) \\ \chi_{2r+2}(z_1) & \omega_{2r+2}(z_2) \end{vmatrix}$$

but $|\, \omega_{2r}(z_1),\ \omega^{2r+2}(z_2) \,|$ is unambiguous. Similarly when the symbol of functionality is tied to its suffix we shall write

$$\left|\, A_r(z_1), B_s(z_2), C_t(z_3) \,\right|^{+} = \begin{vmatrix} A_r(z_1) & B_s(z_1) & C_t(z_1) \\ A_r(z_2) & B_s(z_2) & C_t(z_2) \\ A_r(z_3) & B_s(z_3) & C_t(z_3) \end{vmatrix}.$$

Thus $\left|\, f_t(x),\ p_r(z) \,\right|^{+} = \begin{vmatrix} f_t(x) & p_r(x) \\ f_t(z) & p_r(z) \end{vmatrix}.$

$$
\begin{aligned}
&| \chi_{2s+2}(z),\ \omega_{2s}(z) |^{+} = \prod_{r=1}^{2s+1} b_r \\
&| \chi_{2s+1}(z),\ \omega_{2s+3}(z) |^{+} = z \prod_{r=1}^{2s+2} b_r \\
&| \chi_{2s+4}(z),\ \omega_{2s}(z) |^{+} = (z + b_{2s+3} + b_{2s+4}) \prod_{r=1}^{2s+1} b_r \\
&| \chi_{2s+4}(z),\ \omega_{2s+5}(z) |^{+} = z(z + b_{2s+4} + b_{2s+5}) \prod_{r=1}^{2s+2} b_r .
\end{aligned} \tag{12}
$$

The relation between the even and odd convergents is given by [1]

$$
\begin{aligned}
&| \omega_{2s}(0),\ \chi_{2s+2}(z) |^{+} = \omega_{2s}(0)\chi_{2s+1}(z), \quad \omega_{2s}(0) = \prod_{r=1}^{s} b_{2r}, \\
&| \omega_{2s}(0),\ \omega_{2s+2}(z) |^{+} = \omega_{2s}(0)\omega_{2s+1}(z), \quad \chi_{2s+1}(0) = \prod_{r=0}^{s} b_{2r+1} .
\end{aligned} \tag{13}
$$

From (12) it is easily proved that

$$
\begin{cases}
| \chi_{2r}(z_1),\ \omega_{2r+2}(z_1),\ \omega_{2r+4}(z_2) |^{+} = (z_1 - z_2)\ \omega_{2r+2}(z_2) \prod_{s=1}^{2r+1} b_s \\
| \chi_{2r+1}(z_1),\ \omega_{2r+3}(z_1),\ \omega_{2r+5}(z_2) |^{+} = z_1(z_2 - z_1)\ \omega_{2r+3}(z_2) \prod_{s=1}^{2r+2} b_s .
\end{cases} \tag{14}
$$

2.1 *The orthonormal polynomials.* We introduce the system $\{p_r(x)\}$ where

$$
\int_0^\infty p_r(x)p_s(x)\,d\psi(x) = \delta_{rs}, \tag{15}
$$

with recurrence relation

$$
\begin{aligned}
&p_r(x) = (A_r x - B_r)\ p_{r-1}(x) - C_r p_{r-2}(x),\ p_{-1} = 0, \qquad r = 1, 2, \ldots, \\
&p_0(x) = k_0, \quad p_1(x) = (A_1 x - B_1)k_0
\end{aligned} \tag{16}
$$

and $A_r = k_r/k_{r-1} > 0, \quad C_r = A_r/A_{r-1} > 0,$
where $k_r > 0$ is the highest coefficient in $p_r(x)$. Clearly $B_r > 0$, for

$$
B_r = A_r \int x p^2{}_{r-1}(x) d\psi(x).
$$

Moreover

$$
\begin{cases}
B_r/A_r = b_{2r} + b_{2r-1}, & B_1/A_1 = b_2 \\
A_r^{-2} = b_{2r+1}b_{2r}, & A_1^{-2} = b_3 b_2 \\
k_r^{-2} = \prod_{s=1}^{2r+1} b_s
\end{cases} \tag{17}
$$

---

[1] See J. Shohat, "On Stieltjes Continued Fractions," *American Journal of Math.*, LIV. (1932), 79-84.

by comparison with (11). We also require

$$\left.\begin{aligned} k_r\omega_{2r}(z) &= (-)^r p_r(-z) \\ k_r\chi_{2r}(z) &= (-)^r \int_0^\infty \frac{p_r(-z)-p_r(x)}{z+x}\,d\psi(x) \end{aligned}\right\} \qquad (18)$$

$$r = 0,\ 1,\ 2,\ \ldots.$$

3. *A fundamental identity.* We shall now prove an identity which relates the generalised convergents of

$$F(z_1,\ z_2,\ \ldots\ z_n) = \int_0^\infty \frac{d\psi(x)}{f_n(x)},$$

where $f_n(x) = \prod_{\lambda=1}^{n} (x + z_\lambda)$, to those of $F(z)$.

We consider

$$\Delta_j(x) = \overset{+}{|}\, f_j(x),\ p_r(-z_1),\ p_{r+1}(-z_2),\ \ldots\ p_{r+n-1}(-z_n)\,| \qquad (19)$$

$$= f_n(x) \sum_{s=0}^{r-1} \bar{A}_{sj}(r) q_s(x) \qquad j = l,\ m;\ r = 1,\ 2,\ \ldots\ldots,$$

where

$$f_l(x) = \prod_{\lambda=1}^{l} (x + x_\lambda), \qquad f_m(x) = \prod_{\lambda=1}^{m} (x + y_\lambda), \qquad l < n,\ m \leqslant n,$$

$f_l(x)$, $f_m(x)$, $f_n(x)$ are polynomials in x with real coefficients, $f_n(x) > 0$ for $x \geqslant 0$ with distinct roots, and

$$\int_0^\infty q_r(x) q_s(x) f_n(x) d\psi(x) = 0 \qquad r \neq s$$

$$= \phi_r \qquad r = s.$$

But from Part 1, paragraph 4, we have

$$q_r(x) = (-)^n \frac{k_{r+n}}{k_r} \sum_{s=0}^{r} |\ p_s(-z_1), p_{r+1}(-z_2), \ldots p_{r+n-1}(-z_n)\,|\, p_s(x) \qquad (20)$$

$$\phi_r = (-)^n \frac{k_{r+n}}{k_r} |\ p_r(-z_1), \ldots p_{r+n-1}(-z_n)\,|\,|\ p_{r+1}(-z_1), \ldots\ p_{r+n}(-z_n)\,|. \qquad (21)$$

If now $\psi(x)$ is a solution of a determined S.M.P., and the integrals $\int_0^\infty \frac{(f_l(x))^2}{f_n(x)} d\psi(x)$ and $\int_0^\infty \frac{(f_m(x))^2}{f_n(x)} d\psi(x)$ converge, then Parseval's theorem applies to the functions $f_l(x)/f_n(x)$ and $f_m(x)/f_n(x)$ giving, with respect to the distribution function $\int_0^x f_n(t) d\psi(t)$,

$$F(z_1, z_2, \ldots z_n) = \int_0^\infty \frac{f_l(x) f_m(x)}{f_n(x)}\, d\psi(x) = \sum_{j=0}^{\infty} A_{lj} A_{mj} \phi_j \qquad (22)$$

where

$$A_{kj}\phi_j = \int_0^\infty f_k(x) q_j(x) d\psi(x), \qquad k = l,\ m, \qquad (23)$$

$$j = 0,\ 1,\ 2,\ \ldots.$$

We also write

$$\sum_{j=0}^{r-1} A_{lj}A_{mj}\phi_j = \frac{\chi_r(z_1, z_2, \ldots z_n)}{\omega_r(z_1, z_2, \ldots z_n)} \equiv \frac{\chi_r(z_\lambda)_1^n}{\omega_r(z_\lambda)_1^n} \tag{24}$$

and (24) gives the r^{th} convergent of the n^{th} order C.F. for $F(z_1, z_2, \ldots z_n)$.

But from (19)

$$\int_0^\infty \frac{\Delta_l(x)\Delta_m(x)}{f_n(x)}\, d\psi(x) = \sum_{j=0}^{r-1} \overline{A}_{lj}(r)\overline{A}_{mj}(r)\phi_j, \qquad r = 1, 2, \ldots. \tag{25}$$

where $\overline{A}_{kj}(r) = \mid p_r(-z_1)\ldots.p_{r+n-1}(-z_n) \mid A_{kj}$ $\quad k = l, m,$ $\quad j = 0, 1, \ldots r-1.$

Hence from (24)

$$\frac{\chi_r(z_\lambda)_1^n}{\omega_r(z_\lambda)_1^n} = \mid p_r(-z_1), \ \ldots\ p_{r+n-1}(-z_n) \mid^{-1} \int_0^\infty \frac{\Delta_m(x)f_l(x)}{f_n(x)}\, d\psi(x). \tag{26}$$

But since $l < n$ and the roots of $f_n(x)$ are distinct, we have

$$f_l(x)/f_n(x) = (-)^{n-1} \frac{\overset{+}{\left|z_1^0, z_2^1, \ldots z_{n-1}^{n-2}, \begin{matrix} f_l(-z_n) \\ z_n + x \end{matrix}\right|}}{\left|z_1^0, z_2^1, \ldots z_n^{n-1}\right|}. \tag{27}$$

Using (27) and (18) in (26) we have

$$\frac{\chi_r(z_\lambda)_1^n}{\omega_r(z_\lambda)_1^n} = \begin{vmatrix} g & f_m(-z_1) & f_m(-z_2) & \ldots & f_m(-z_n) \\ g_r & \omega_{2r}(z_1) & \omega_{2r}(z_2) & \ldots & \omega_{2r}(z_n) \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ \cdot & \cdot & & & \\ g_{r+n-1} & \omega_{2r+2n-2}(z_1) & \omega_{2r+2n-2}(z_2) & \ldots & \omega_{2r+2n-2}(z_n) \end{vmatrix} \tag{28}$$

where

$$\begin{cases} g = (-)^n \overset{+}{\left|z_1^0, z_2^1, \ldots z_{n-1}^{n-2}, F_m(z_n)f_l(-z_n)\right|} \Big/ \left|z_1^0, z_2^1, \ldots z_n^{n-1}\right| \\ g_s = (-)^n \overset{+}{\left|z_1^0, z_2^1, \ldots z_{n-1}^{n-2}, \chi_{2s}(z_n)f_l(-z_n)\right|} \Big/ \left|z_1^0, z_2^1, \ldots z_n^{n-1}\right|, \\ \qquad\qquad s = r, r+1, \ldots r+n-1 \\ F_m(z_s) = \int_0^\infty \frac{f_m(-z_s) - f_m(x)}{z_s + x}\, d\psi(x), \qquad s = 1, 2, \ldots n \end{cases}$$

and (28) consists of the ratio of the determinant of order $n+1$ and the determinant obtained from this by deleting its first row and column. An alternative to (28) appears by using the partial fraction

form of the elements in the first column, and we find

$$\frac{\chi_r(z_\lambda)_1^n}{\omega_r(z_\lambda)_1^n} = -\begin{vmatrix} \sum_{s=1}^{n} \frac{F_m(z_s)f_l(-z_s)}{\prod_{r=1}^{n}{}' (z_r - z_s)} & f_m(-z_1) & f_m(-z_2) & \dots & f_m(-z_n) \\ \sum_{s=1}^{n} \frac{\chi_{2r}(z_s)f_l(-z_s)}{\prod_{r=1}^{n}{}' (z_r - z_s)} & \omega_{2r}(z_1) & \omega_{2r}(z_2) & \dots & \omega_{2r}(z_n) \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \sum_{s=1}^{n} \frac{\chi_{2r+2n-2}(z_s)f_l(-z_s)}{\prod_{r=1}^{n}{}' (z_r - z_s)} & \omega_{2r+2n-2}(z_1) & \omega_{2r+2n-2}(z_2) & \dots & \omega_{2r+2n-2}(z_n) \end{vmatrix} \tag{29}$$

If we take $\omega_r(z_\lambda)_1^n = |\ \omega_{2r}(z_1), \dots \omega_{2r+2n-2}(z_n)\ |\ /\ |\ z_1^0, z_2^1, \dots z_n^{n-1}\ |$ then (28) and (29) give expressions for $\chi_r(z_\lambda)_1^n$ in terms of $\chi_{2s}(z_\lambda)$ and $\omega_{2s}(z_\lambda)$, $s = r$ to $r+n-1$, $\lambda = 1$ to n. We note that if in addition to $l < n$, we have $m < n$, then l and m may be interchanged in (28) and (29) yielding an identity between two forms of the numerator $\chi_r(z_\lambda)_1^n$.

The confluent case of (28) — (29) in its general form is complicated, but particular cases may be obtained from first principles. Thus if $z_1 = z_2 \dots = z_j$, $j \leqslant n$, then the limiting form appears by letting $z_2 \to z_1$, $z_3 \to z_1$, $\dots\dots$ $z_j \to z_1$ in succession, and subtracting appropriate columns. The complication arises from the fact that $f_l(x)$ and $f_m(x)$ may be functions of z_λ, $\lambda = 1$ to n.

4.0 We shall now consider generalised continued fraction expansions for $F(z_1, z_2, \dots z_n)$ of four kinds, namely

(i) convergent increasing sequences,

(ii) convergent decreasing sequences,

(iii) convergent sequences,

(iv) convergent sequences involving an arbitrary parameter.

In the main we shall confine our attention to second order C.F.'s for $F(z_1, z_2)$. We assume that $\psi(x)$ is the unique solution of a S.M.P.

4.1 *Increasing sequences.* In (29) take $f_l = f_m = 1$, $f_n(x) = (x + z_1)(x + z_2) > 0$, $z_1 \neq z_2$ ($z_2 = \bar{z}_1$ if z_1 is complex) so that $F_m(z_s) = 0$, and

$$F(z_1, z_2) = \int_0^\infty \frac{d\psi(x)}{(x + z_1)(x + z_2)}$$

$$= \lim_{r\to\infty} -\frac{1}{z_1 - z_2} \begin{vmatrix} 0 & 1 & 1 \\ \chi_{2r}(z_2) - \chi_{2r}(z_1) & \omega_{2r}(z_1) & \omega_{2r}(z_2) \\ \chi_{2r+2}(z_2) - \chi_{2r+2}(z_1) & \omega_{2r+2}(z_1) & \omega_{2r+2}(z_2) \end{vmatrix}$$

where the expansion, in view of (22), is an increasing sequence. By (12) this may be written

$$F(z_1, z_2) = \text{l.i.s.}_{r\to\infty} \frac{|\chi_{2r}(z_1),\ \omega_{2r+2}(z_2)|^+ + |\chi_{2r}(z_2),\ \omega_{2r+2}(z_1)|^+ + 2\prod_{s=1}^{2r+1} b_s}{(z_1 - z_2)\,|\,\omega_{2r}(z_2),\ \omega_{2r+2}(z_1)\,|} \tag{30}$$

where we use the abbreviation l.i.s. for *limit of the increasing sequence.*[1]

If in particular $z_1 = z_2 = z$ then by letting $z_2 \to z_1$ in (30) we have

$$F(z, z) = \int_0^\infty \frac{d\psi(x)}{(z + x)^2} = \text{l.i.s.}_{r\to\infty} \frac{|\,\omega'_{2r}(z),\ \chi'_{2r+2}(z)|^+}{|\,\omega_{2r}(z),\ \omega'_{2r+2}(z)\,|} \tag{31}$$

$$= -\frac{dF(z)}{dz}, \qquad z > 0.$$

The general formula of this type is found similarly from (28) and gives

$$\int_0^\infty \frac{d\psi(x)}{(z + x)^{n+1}} = \text{l.i.s.}_{r\to\infty} \frac{(-)^n}{n!} \frac{|\,\chi^{(n)}_{2r}(z), \omega^{(1)}_{2r+2}(z), \omega^{(2)}_{2r+4}(z), \ldots \omega^{(n)}_{2r+n}(z)|^+}{|\,\omega_{2r}(z), \omega^{(1)}_{2r+2}(z), \omega^{(2)}_{2r+4}(z) \ldots \omega^{(n)}_{2r+2n}(z)|^+} \tag{32}$$

$$z > 0.$$

---

[1] As a particular example suppose that by using (4) and an equivalence transformation we find the convergent expansion

$$\tfrac{1}{2}\int_0^\infty \frac{xe^{-\sqrt{x}}dx}{x + z} = \lim_{r\to\infty} \frac{\chi_r(z)}{\omega_r(z)} = \frac{b_1}{z} + \frac{b_2}{1} + \frac{b_3}{z} + \frac{b_4}{1} + \ldots .$$

$$\sim \frac{3!}{z} - \frac{5!}{z^2} + \frac{7!}{z^3} - \frac{9!}{z^4} + \ldots .$$

Then by (30) with $z_1 = it = -z_2$, $t > 0$, we have a convergent expansion for

$$F(it, -it) = \tfrac{1}{2}\int_0^\infty \frac{xe^{-\sqrt{x}}dx}{x^2 + t^2} \sim \frac{3!}{t^2} - \frac{7!}{t^4} + \frac{11!}{t^6} - \frac{15!}{t^8} + \ldots$$

But the Stieltjes C.F for $F(it, -it) = \frac{1}{4}\int_0^\infty \frac{\exp^{-x^{\frac{1}{2}}}dx}{x + t^2}$ diverges by oscillation.

There are of course other forms of increasing sequences for $F(z_1, z_2)$; for example we could use $f_l = x$, $f_m = x$, $f_n = x^2(x+z_1)(x+z_2)$ in (28). But (30) seems to be the simplest of this type.

4. 2 *Decreasing sequences.* When the roots z_1, z_2 are distinct and $(x+z_1)(x+z_2) = x^2 + 2px + q$, $q - p^2 > 0$, then we use the relation

$$(q-p^2)F(z_1, z_2) = b_1 - \int_0^\infty \frac{(x+p)^2 d\psi(x)}{(x+z_1)(x+z_2)}. \tag{33}$$

Taking in (29) $f_l = f_m = x + p$, $f_n = (x+z_1)(x+z_2)$, $F_l(z_s) = -b_1$, we have

$$(q-p^2)\ F(z_1, z_2) = b_1 - \underset{r\to\infty}{\text{l.i.s.}} \begin{vmatrix} b_1 & \tfrac{1}{2}(z_2 - z_1) & \tfrac{1}{2}(z_1 - z_2) \\ -\tfrac{1}{2}(\chi_{2r}(z_1) + \chi_{2r}(z_2)) & \omega_{2r}(z_1) & \omega_{2r}(z_2) \\ -\tfrac{1}{2}(\chi_{2r+2}(z_1) + \chi_{2r+2}(z_2)) & \omega_{2r+2}(z_1) & \omega_{2r+2}(z_2) \end{vmatrix}$$

and after using (12) this leads to [1]

$$F(z_1, z_2) = \underset{r\to\infty}{\text{l.d.s}} \frac{|\chi_{2r}(z_1),\ \omega_{2r+2}(z_2)|^+ + |\chi_{2r}(z_2),\ \omega_{2r+2}(z_1)|^+ - 2\prod_{s=1}^{2r+1} b_s}{(z_1 - z_2)\ |\ \omega_{2r}(z_2),\ \omega_{2r+2}(z_1)\ |}. \tag{34}$$

When $q - p^2 > 0$ it will be seen that the difference between corresponding convergents of (30) and (34) is

$$\frac{-4\prod_{s=1}^{2r+1} b_s}{(z_1 - z_2)\ |\ \omega_{2r}(z_2),\ \omega_{2r+2}(z_1)\ |} \tag{35}$$

and this exceeds the absolute error in either of them.

Again, taking $f_l = f_m = (x+p)^2$, $f_n = (x+z_1)(x+z_2)(x+p)^2$, $q - p^2 > 0$ in (28) (taking the limiting form with $z_3 = z_4 = p$), we find

$$F(z_1, z_2) = \underset{r\to\infty}{\text{l.d.s.}} \begin{vmatrix} 0 & 1 & 1 & 0 & 0 \\ \dfrac{\chi_{2r}(z_1) - \chi_{2r}(z_2)}{z_1 - z_2} & \omega_{2r}(z_1) & \omega_{2r}(z_2) & \omega_{2r}(p) & \omega'_{2r}(p) \\ \dfrac{\chi_{2r+2}(z_1) - \chi_{2r+2}(z_2)}{z_1 - z_2} & \omega_{2r+2}(z_1) & \omega_{2r+2}(z_2) & \omega_{2r+2}(p) & \omega'_{2r+2}(p) \\ \dfrac{\chi_{2r+4}(z_1) - \chi_{2r+4}(z_2)}{z_1 - z_2} & \omega_{2r+4}(z_1) & \omega_{2r+4}(z_2) & \omega_{2r+4}(p) & \omega'_{2r+4}(p) \\ \dfrac{\chi_{2r+6}(z_1) - \chi_{2r+6}(z_2)}{z_1 - z_2} & \omega_{2r+6}(z_1) & \omega_{2r+6}(z_2) & \omega_{2r+6}(p) & \omega'_{2r+6}(p) \end{vmatrix} \tag{36}$$

[1] l.d.s. means *limit of the decreasing sequence.*

In particular if $p=0$, $q>0$, then after using (12) and (13) we find

$$\int_0^\infty \frac{d\psi(x)}{x^2+q}$$

$$= \underset{r\to\infty}{\text{l.d.s.}} \frac{|\chi_{2r+1}(z_1), \omega_{2r+3}(z_2), \omega'_{2r+5}(0)|^{+} + |\chi_{2r+1}(z_2), \omega_{2r+3}(z_1), \omega'_{2r+5}(0)|^{+} - 2q\omega'_{2r+3}(0)\prod_{s=1}^{2r+2} b_s}{(z_2-z_1)\,|\omega_{2r+1}(z_1), \omega_{2r+3}(z_2), \omega'_{2r+5}(0)|} \qquad (37)$$

Another interesting possibility is to use the expression

$$qF(z_1, z_2) = b_1 - \int_0^\infty \frac{x(x+2p)d\psi(x)}{(x+z_1)(x+z_2)}, \qquad q>0, \qquad (38)$$

where $p>0$, $(x+z_1)(x+z_2) = x^2+2px+q$, taking $f_l = f_m = x(x+2p)$, $f_n = x(x+2p)(x^2+2px+q)$ in (29) with $z_1, z_2, z_3 = 2p$, $z_4 = 0$ and $F_l(z_s) = (z_s-2p)b_1 - a_1$

$$\sum_{s=1}^{4} \frac{F_l(z_s)f_m(-z_s)}{\prod'_{r=1}^{4}(z_r-z_s)} = -b_1, \qquad \sum_{s=1}^{4} \frac{\chi(z_s)f_m(-z_s)}{\prod'_{r=1}^{4}(z_r-z_s)} = \frac{\chi(z_1)-\chi(z_2)}{z_2-z_1}.$$

Using (12) and (13) we find after some simplification

$$F(z_1, z_2) = \underset{r\to\infty}{\text{l.d.s.}} \frac{|\chi_{2r+1}(z_1), \omega_{2r+3}(z_2), \omega_{2r+5}(2p)|^{+} + |\chi_{2r+1}(z_2), \omega_{2r+3}(z_1), \omega_{2r+5}(2p)|^{+} - 2q\omega_{2r+3}(2p)\prod_{s=1}^{2r+2} b_s}{(z_2-z_1)\,|\omega_{2r+1}(z_1), \omega_{2r+3}(z_2), \omega_{2r+5}(2p)|} \qquad (39).$$

By setting $z_2 = z_1 + h$, $h\to 0$, we have

$$\int_0^\infty \frac{d\psi(x)}{(x+z)^2} = \underset{r\to\infty}{\text{l.d.s.}} - \frac{|\chi'_{2r+1}(z), \omega'_{2r+3}(z), \omega_{2r+5}(2z)|^{+}}{|\omega_{2r+1}(z), \omega'_{2r+3}(z), \omega'_{2r+5}(2z)|}, \qquad z>0. \quad (40)$$

4.3 *Convergent sequences.* The approximations considered in 4.1 and 4.2 provide lower and upper bounds, but there are other approximations which merely converge. We shall briefly consider four simple types, derived from (28) — (29).

(i) $f_l = x$, $f_m = 1$, $f_n = x(x+z_1)(x+z_2)$, $z_1 \neq z_2$, $z_1, z_2, > 0$.

Then $F(z_1, z_2) =$

$$\lim_{r\to\infty} \frac{|\chi_{2r+1}(z_1), \omega_{2r+3}(z_2)|^{+} + |\chi_{2r+1}(z_2), \omega_{2r+3}(z_1)|^{+} - (z_1+z_2)\prod_{1}^{2r+2} b_s - (z_1-z_2)(\omega_{2r+2}(z_1)-\omega_{2r+2}(z_2))\prod_{0}^{r} b_{2s+1}}{(z_1-z_2)\,|\omega_{2r+1}(z_2), \omega_{2r+3}(z_1)|} \qquad (41)$$

In particular if $z_1 = z_2$, using (12) we find

$$F(z, z) = \lim_{r\to\infty} \frac{|\ \chi'_{2r+1}(z),\ \omega'_{2r+3}(z)\ |^{+} + \omega'_{2r+2}(z) \prod_{0}^{r} b_{2s+1}}{|\ \omega'_{2r+1}(z),\ \omega_{2r+3}(z)\ |}. \qquad (42)$$

(ii) $q - p^2 > 0$. In (33) take $f_m = (x+p)^2$, $f_l = 1$, $f_n = x^2 + 2px + q$, and we find an expression which is exactly the same as (30). This brings to light an interesting identity, for we have the two expansions

$$\int_0^\infty \frac{(x+p)^2 d\psi(x)}{x^2+2px+q} \underset{r\to\infty}{=} \text{l.i.s.} \left\{ b_1 - \tfrac{1}{4}(z_1 - z_2) \frac{|\ \chi_{2r}(z_1),\ \omega_{2r+2}(z_2)\ |^{+} + |\ \chi_{2r}(z_2),\ \omega_{2r+2}(z_1)\ |^{+} - 2\prod_{1}^{2r+1} b_s}{|\ \omega_{2r}(z_1),\ \omega_{2r+2}(z_2)\ |} \right\}$$

$$= \lim_{r\to\infty} \left\{ b_1 - \tfrac{1}{4}(z_1 - z_2) \frac{|\ \chi_{2r}(z_1),\ \omega_{2r+2}(z_2)\ |^{+} + |\ \chi_{2r}(z_2),\ \omega_{2r+2}(z_1)\ |^{+} + 2\prod_{1}^{2r+1} b_s}{|\ \omega_{2r}(z_1),\ \omega_{2r+2}(z_2)\ |} \right\}$$

$$q - p^2 > 0 \text{ or } z_1, z_2 > 0, (z_1 \neq z_2). \qquad (43)$$

The difference between corresponding convergents of the two expansions in (43) is therefore $(z_1 - z_2) \prod_{1}^{2r+1} b_s / |\ \omega_{2r}(z_1),\ \omega_{2r+2}(z_2)\ |$. In terms of the persymmetric determinants[1] mentioned in Part 3, 3 (*a*) this comes to

$$\begin{vmatrix} 0 & \mu_0^{(1)} & \mu_1^{(1)} & \cdots & \mu_r^{(1)} \\ \mu_0^{(1)} & \bar{\mu}_0 & \bar{\mu}_1 & \cdots & \bar{\mu}_r \\ \mu_1^{(1)} & \bar{\mu}_1 & \bar{\mu}_2 & \cdots & \bar{\mu}_{r+1} \\ \vdots & \vdots & \vdots & & \vdots \\ \mu_r^{(1)} & \bar{\mu}_r & \bar{\mu}_{r+1} & \cdots & \bar{\mu}_{2r} \end{vmatrix} - \begin{vmatrix} 0 & \mu_0^{(2)} & \mu_1^{(2)} & \cdots & \mu_r^{(2)} \\ \mu_0^{(0)} & \bar{\mu}_0 & \bar{\mu}_1 & \cdots & \bar{\mu}_r \\ \mu_1^{(0)} & \bar{\mu}_1 & \bar{\mu}_2 & \cdots & \bar{\mu}_{r+1} \\ \vdots & \vdots & \vdots & & \vdots \\ \mu_r^{(0)} & \bar{\mu}_r & \bar{\mu}_{r+1} & \cdots & \bar{\mu}_{2r} \end{vmatrix} = \prod_{0}^{r+1} k_s^{-2} \qquad (44)$$

where

$$\mu_s^{(0)} = \int_0^\infty x^s d\psi(x), \qquad \mu_s^{(1)} = \int_0^\infty (x+p) x^s d\psi(x),$$

$$\mu_s^{(2)} = \int_0^\infty (x+p)^2 x^s d\psi(x), \qquad \bar{\mu}_s = \int_0^\infty (x^2+2px+q) x^s d\psi(x)$$

$$k_s^{-2} = \prod_{1}^{2s+1} b_r$$

and $x^2 + 2px + q$ is non-negative for $0 \leqq x < \infty$.

[1] There is a similar identity for the diagonal determinants given in Part 3, 3 (*b*).

(iii) If we use (38) with $f_l = x$. $f_m = x + 2p$, $f_n = (x + z_1)(x + z_2)$ there follows the expansion

$$F(z_1,\ z_2) = \lim_{r\to\infty} \frac{|\ \chi_{2r}(z_1),\ \omega_{2r+2}(z_2)\ |^{+} + |\ \chi_{2r}(z_2),\ \omega_{2r+2}(z_1)\ |^{+} + (z_1 z_2)^{-1}(z_1^2 + z_2^2)\ \prod_1^{2r+1} b_s}{(z_1 - z_2)\ |\ \omega_{2r}(z_2),\ \omega_{2r+2}(z_1)\ |} \tag{45}$$

$$z_1,\ z_2 > 0, \qquad z_1 \neq z_2 \qquad \text{or} \qquad q - p^2 > 0,$$

and in particular

$$F(z,\ z) = \lim_{r\to\infty} \frac{|\ \omega'_{2r}(z),\ \chi'_{2r+2}(z)\ |^{+} + z^{-2} \prod_1^{2r+1} b^s}{|\ \omega'_{2r}(z),\ \omega_{2r+2}(z)\ |}. \tag{46}$$

(iv) Using (38) with $f_l = x$, $f_m = x\,(x + 2p)$, $f_n = x\,(x^2 + 2px + q)$ we find

$$F(z_1,\ z_2) = \lim_{r\to\infty} \frac{|\ \chi_{2r+1}(z_1),\ \omega_{2r+3}(z_2)\ |^{+} + |\ \chi_{2r+1}(z_2),\ \omega_{2r+3}(z_1)\ |^{+} - (z_1 + z_2) \prod_1^{2r+2} b_s}{(z_1 - z_2)\ |\ \omega_{2r+1}(z_2),\ \omega_{2r+3}(z_1)\ |} \tag{47}$$

$$x^2 + 2px + q > 0 \text{ for } x \geqq 0$$

and in particular

$$F(z,\ z) = \lim_{r\to\infty} -\frac{|\ \chi'_{2r+1}(z),\ \omega'_{2r+3}(z)\ |^{+}}{|\ \omega_{2r+1}(z),\ \omega'_{2r+3}(z)\ |}, \qquad z > 0. \tag{48}$$

4.4 *Expansions with arbitrary parameters.* An unusual type of expansion appears if in (29) we take $f_l = f_m = x + p$, $f_n = (x+z)(x+p)^2$ where we assume $z > 0$, and p real. Then

$$F(z) = \int_0^\infty \frac{d\psi(x)}{z + x} = \underset{r\to\infty}{\text{l.i.s.}} \frac{|\ \chi_{2r}(z) - \chi_{2r}(p),\ \omega_{2r+2}(p),\ \omega'_{2r+4}(p)\ |^{+} + k_r^{-2}(\omega_{2r+2}(p) - \omega_{2r+2}(z))}{|\ \omega_{2r}(z),\ \omega_{2r+2}(p),\ \omega'_{2r+4}(p)\ |} \tag{49}$$

for all real $p \neq z$.

If $p = z$ then

$$F(z) = \underset{r\to\infty}{\text{l.i.s.}} \frac{|\ \omega_{2r}(z),\ \omega'_{2r+2}(z),\ \chi''_{2r+4}(z)\ |^{+} - k_r^{-2}\omega''_{2r+2}(z)}{|\ \omega_{2r}(z),\ \omega'_{2r+2}(z),\ \omega''_{2r+4}(z)\ |}, \qquad z > 0. \tag{50}$$

Similarly from $f_l = f_m = x\,(x + p)$, $f_n = x(x + z)\,(x + p)^2$ we find

$$F(z) = \underset{r\to\infty}{\text{l.d.s.}} \frac{|\ \chi_{2r+1}(z) - \chi_{2r+1}(p),\ \omega_{2r+3}(p),\ \omega'_{2r+5}(p)\ |^{+} - pz^{-1}(z\omega_{2r+3}(p) - p\omega_{2r+3}(z)) \prod_1^{2r+2} b_s}{|\ \omega_{2r+1}(z),\ \omega_{2r+3}(p),\ \omega'_{2r+5}(p)\ |} \tag{51}$$

$z > 0$, $p \neq 0$, $p \neq z$.

In particular if $p = 0$ then for $z > 0$

$$F(z) = \underset{r\to\infty}{\text{l.d.s.}} \frac{|\chi_{2r+1}(z)-\chi_{2r+1}(0),\ \omega'_{2r+3}(0),\ \omega''_{2r+5}(0)|^{+} - 2z^{-1}(z\omega'_{2r+3}(0)-\omega_{2r+3}(z)) \prod_{1}^{2r+2} b_s}{|\omega_{2r+1}(z),\ \omega'_{2r+3}(0),\ \omega''_{2r+5}(0)|} \tag{52}$$

and if $p = z > 0$ then

$$F(z) = \underset{r\to\infty}{\text{l.d.s.}} \frac{|\omega_{2r+1}(z),\ \chi'_{2r+3}(z),\ \omega''_{2r+5}(z)|^{+} + 2(z^{-1}\omega_{2r+3}(z)-\omega'_{2r+3}(z)) \prod_{1}^{2r+2} b_s}{|\omega_{2r+1}(z),\ \omega'_{2r+3}(z),\ \omega''_{2r+5}(z)|}. \tag{53}$$

Calling the rth convergents of (49) and (51) $g_r(z, p)$ and $\hat{g}_r(z, p)$ respectively, we may consider the sequences $\{\max_p g_r(z, p)\}$ and $\{\min_p \hat{g}^r(z, p)\}$ as approximations to $F(z)$, assuming that stationary values exist. That such values do exist is seen from the following asymptotic expansions:

$$g_r(z, p) = \frac{\chi_{2r}(z)}{\omega_{2r}(z)} - \frac{2\prod_{1}^{2r+1} b_s}{p\omega^2_{2r}(z)} + o(p^{-1}) \text{ as } |p| \to \infty \tag{54}$$

$$\hat{g}_r(z, p) = \frac{\chi_{2r+1}(z)}{\omega_{2r+1}(z)} + \frac{2z}{p}\frac{\prod_{1}^{2r+2} b_s}{\omega^2_{2r+1}(z)} + o(p^{-1}) \text{ as } |p| \to \infty. \tag{55}$$

It is evident from (54) that for p large and negative $g_r(z, p)$ is a closer approximation to $F(z)$ than $\chi_{2r}(z)/\omega_{2r}(z)$; similarly for $\hat{g}_r(z, p)$. We shall return to a consideration of these approximations in a later part.

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A Determinantal Expansion for a Class of Definite Integral

Part 5. Recurrence Relations

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1. We develop here the recurrence relations for the generalised C.F.'s introduced in Part 3 (Shenton[1] 1956). In the main the discussion will be limited to second order C.F.'s, but results for higher orders will be given when these are not complicated.

We shall give three forms of recurrence relation, one involving *recurrent* determinants, and another corresponding to the *even* and *odd* parts of a Stieltjes C.F. In addition we shall show how to write down directly the recurrence relations for a second order C.F. being given the first order C.F. Several numerical examples are given in illustration.

2.0. We consider the C.F. "corresponding"[2] to a determined Stieltjes moment problem, and write

$$F(z) = \int_0^\infty \frac{d\psi(x)}{x+z} = \frac{b_1}{z} + \frac{b_2}{1} + \frac{b_3}{z} + \frac{b_4}{1} + \ldots, \quad z > 0, \tag{1}$$

and for the "contracted" form

$$F(z) = \frac{a_0}{z+c_1} - \frac{a_1}{z+c_2} - \frac{a_2}{z+c_3} - \ldots \tag{2}$$

where $a_s = b_{2s}\, b_{2s-1}$, $s > 0$; $a_0 = b_1$,

$c_s = b^*_{2s-1} + b_{2s}$, $s > 1$; $b^*_s = b_s$, $s > 1$, $b^*_1 = 0$.

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[1] We shall refer to the previous four papers on this subject as **§1**, **§2**, **§3**, and **§4**. respectively.

[2] Stieltjes preferred to write the "corresponding" C.F. in the form

$$F(z) = \frac{1}{a_1 z} + \frac{1}{a_2} + \frac{1}{a_3 z} + \frac{1}{a_4} + \ldots,$$

in which case the Stieltjes moment problem is determined if $\sum_1^\infty a_s$ diverges, the a's being positive.

The s^{th} convergent of (1) will be written $\chi_s(z)/w_s(z)$, and that of the *even* part $\chi_{2s}(z)/w_{2s}(z)$, where $\chi_0(z) = 0$, $\chi_1(z) = b_1$, $w_0(z) = 1$, $w_1(z) = z$. In the notation of §3 the expression (2) becomes

$$F(z) = \underset{s\to\infty}{\text{l. i. s.}} \frac{a_0 K_{s-1}(\beta_1, a_1)}{K_s(\beta_0, a_0)} \tag{3}$$

where $\beta_s = z + c_{s+1}$, $a_s = \sqrt{a_{s+1}}$ and $K_s(\beta_0, a_0)$ is a continuant determinant of order s with elements $\beta_0, \beta_1, \ldots$ along the diagonal through (1, 1) and elements $a_0, a_1, \ldots$ along the diagonals through (2, 1) and (1, 2).

The second order C.F. can be written[1]

$$F(z_1, z_2) = \int_0^\infty \frac{d\psi(x)}{(x+z_1)(x+z_2)} = \underset{s\to\infty}{\text{l. i. s.}}\ a_0 \frac{K_{s-1}(\gamma_1, \beta_1, a_1)}{K_s(\gamma_0, \beta_0, a_0)} \tag{4a}$$

where
$$a_s = \sqrt{(a_{s+1}\, a_{s+2})}, \tag{4b}$$
$$\beta_s = (p + c_{s+1} + c_{s+2})\sqrt{a_{s+1}},$$
$$\gamma_s = q + pc_{s+1} + c_{s+1}^2 + a_s^* + a_{s+1},$$
$$a_s^* = a_s,\ s > 0, \qquad a_0^* = 0,$$
$$(x+z_1)(x+z_2) \equiv x^2 + px + q > 0,\ x \geqslant 0.$$

Similarly the third order C.F. may be expressed as

$$F(z_1, z_2, z_3) = \int_0^\infty \frac{d\psi(x)}{(x+z_1)(x+z_2)(x+z_3)} = \underset{s\to\infty}{\text{l. i. s.}}\ a_0 \frac{K_{s-1}(\delta_1, \gamma_1, \beta_1, a_1)}{K_s(\delta_0, \gamma_0, \beta_0, a_0)} \tag{5a}$$

where
$$a_s = \sqrt{(a_{s+1}\, a_{s+2}\, a_{s+3})},$$
$$\beta_s = (p + c_{s+1} + c_{s+2} + c_{s+3})\sqrt{(a_{s+1}\, a_{s+2})}, \tag{5b}$$
$$\gamma_s = \{q + p(c_{s+1} + c_{s+2}) + c_{s+2}^2 + c_{s+2}c_{s+1} + c_{s+1}^2 + a_{s+2} + a_{s+1} + a_s^*\}\sqrt{a_{s+1}},$$
$$\delta_s = r + qc_{s+1} + pc_{s+1}^2 + c_{s+1}^3 + p(a_{s+1} + a_s^*) + a_{s+1}c_{s+2} + 2a_{s+1}c_{s+1} + 2a_s^* c_{s+1} + a_s^* c_s,$$
$$a_s^* = a_s,\ s > 0, \qquad a_0^* = 0,$$
$$(x+z_1)(x+z_2)(x+z_3) \equiv x^3 + px^2 + qx + r > 0,\ x \geqslant 0.$$

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[1] $K_s(\gamma_0, \beta_0, a_0)$ is a determinant of order s with elements $\gamma_0, \gamma_1, \ldots$ along the diagonal through (1, 1), $\beta_0, \beta_1 \ldots$ along the diagonals through (2, 1) and (1, 2), and $a_0, a_1, \ldots$ along the diagonals through (3, 1) and (1, 3). The determinant $K_s(\gamma_0, \beta_0, a_0)$ is symmetric with elements in five diagonals only, and may be regarded as a form of generalised continuant. The extension of the notation is obvious.

2.1 To develop the numerators and denominators of (4a) and (5a) in terms of recurrent determinants we require the following lemma.

LEMMA. *If* $K_s\left(h_1, \begin{matrix} g_1, f_1 \\ g_1^1, f_1^1 \end{matrix}\right)$ *is a determinant of order* s *with elements* $f_1, f_2, \ldots$ *along the diagonal through* (1, 3), $g_1, g_2, \ldots$ *along the diagonal through* (1, 2), $h_1, h_2, \ldots$ *along the diagonal through* (1, 1) *and so on, then* [1]

$$K_{s-2}\left(g_1, \begin{matrix} f_1 \\ h_2, g_2^1, f_2^1 \end{matrix}\right) K_s\left(g_0, \begin{matrix} f_0 \\ h_1, g_1^1, f_1^1 \end{matrix}\right) = \begin{vmatrix} K_{s-1}\left(g_1, \begin{matrix} f_1 \\ h_2, g_2^1, f_2^1 \end{matrix}\right) & K_{s-1}\left(h_1, \begin{matrix} g_1, f_1 \\ g_1^1, f_1^1 \end{matrix}\right) \\ \prod_{\lambda=0}^{s-2} f_\lambda & K_{s-1}\left(g_0, \begin{matrix} f_0 \\ h_1, g_1^1, f_1^1 \end{matrix}\right) \end{vmatrix}.$$

The proof is straightforward. For consider $K_s\left(g_0, \begin{matrix} f_0 \\ h_1, g_1^1, f_1^1 \end{matrix}\right)$. Delete the first and last rows and columns, and use the remaining array as a pivot.[2] The result then follows. We now deduce that

$$\prod_{\lambda=0}^{s-2} f_\lambda \, K_{s-1}\left(h_1, \begin{matrix} g_1, f_1 \\ g_1^1, f_1^1 \end{matrix}\right) = \begin{vmatrix} K_{s-1}\left(g_1, \begin{matrix} f_1 \\ h_2, g_2^1, f_2^1 \end{matrix}\right) & K_{s-2}\left(g_1, \begin{matrix} f_1 \\ h_2, g_2^1, f_2^1 \end{matrix}\right) \\ K_s\left(g_0, \begin{matrix} f_0 \\ h_1, g_1^1, f_1^1 \end{matrix}\right) & K_{s-1}\left(g_0, \begin{matrix} f_0 \\ h_1, g_1^1, f_1^1 \end{matrix}\right) \end{vmatrix}. \tag{6}$$

Applying (6) to the numerator and denominator of (4a) we find

$$F(z_1, z_2) = \underset{s \to \infty}{\text{l. i. s.}} \; a_0 \frac{| V_s, W_{s+1} |}{| U_s, V_{s+1} |}, \tag{7}$$

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[1] A determinant of the form $K_s\left(g_0, \begin{matrix} f_0 \\ h_1, g_1^1, f_1^1 \end{matrix}\right)$ of order s, with elements $f_0, f_1, \ldots$ along the first superdiagonal, $g_0, g_1, \ldots$ along the principal diagonal, $h_1, h_2, \ldots$ along the first subdiagonal, and so on, will be referred to as a *recurrent determinant*, or simply a *recurrent*. By expanding a *recurrent* of this form by its last row, it will be seen that it follows a fourth order recurrence relation. Similarly a *recurrent* with n subdiagonals may be shown to follow a *recurrence* relation of order $n + 1$.

[2] See for example A. C. Aitken, *Determinants and Matrices* (fourth edition, Edinburgh, 1946,), pp. 48-49.

where $U_s = K_s\left(\beta^*_{-1}, \begin{matrix} a^*_{-1} \\ \gamma_0, \beta_0, a_0 \end{matrix}\right)$

$V_s = K_{s-1}\left(\beta_0, \begin{matrix} a_0 \\ \gamma_1, \beta_1, a_1 \end{matrix}\right)$

$W_s = K_{s-2}\left(\beta_1, \begin{matrix} a_1 \\ \gamma_2, \beta_2, a_2 \end{matrix}\right),$

$\beta^*_s = \beta_s,\ a^*_s = a_s,\ s \geqslant 0; \qquad \beta^*_{-1} = 0, \qquad a^*_{-1} = 1,$

and the *recurrents* U_s, V_s, W_s follow the relation

$$y_s = \beta_{s-2}y_{s-1} - \gamma_{s-2}a^*_{s-3}y_{s-2} + \beta_{s-3}a_{s-3}a_{s-4}y_{s-3} - a_{s-3}a^2_{s-4}a_{s-5}y_{s-4}, \tag{8}$$

with $U_0 = 1, \quad U_1 = 0, \quad U_s = 0, \quad s < 0,$

$V_1 = 1, \quad V_2 = \beta_0, \quad V_s = 0, \quad s < 1,$

$W_2 = 1, \quad W_3 = \beta_1, \quad W_s = 0, \quad s < 2,$

the values of a_s, β_s, γ_s being given in (4b).

The s^{th} approximant to $F(z_1, z_2)$ depends upon the six terms U_s, U_{s+1}, V_s, V_{s+1}, W_s, W_{s+1}, each of which follows a recurrence relation of order four. Hence to advance the approximation process one stage, it is necessary to evaluate a value of each of U_s, V_s, W_s, and this will involve twelve calculations. We shall show in a later section that $| V_s, W_{s+1} |$ and $| U_s, V_{s+1} |$ (or equivalent expressions) follow recurrence relations of order five, so that there is perhaps an economy to be gained by this method.

A decreasing sequence of *upper bounds* may be derived from the expression.

$$F(z_1, z_2) = (q - \tfrac{1}{4}p^2)^{-1}a_0 - (q - \tfrac{1}{4}p^2)^{-1}\int_0^\infty \frac{(x + \tfrac{1}{2}p)^2\, d\psi(x)}{x^2 + px + q}, \tag{9}$$

and it is not difficult to show that the difference between the s^{th} approximations that arise from (9) and (4a) is

$$\frac{(q - \tfrac{1}{4}p^2)^{-1} \prod_{\lambda=0}^{s} a_\lambda}{K_s(\gamma_0, \beta_0, a_0)}, \tag{10}$$

it being assumed that $q - \tfrac{1}{4}p^2 > 0$.

2.2 For third order C.F's we require the following extension of the lemma:

$$Q_s^{(0)} K_s\left(\begin{matrix} i_1, & h_1, & g_1, & f_1 \\ & h_1^1, & g_1^1, & f_1^1 \end{matrix}\right) = \begin{vmatrix} T_s^{(1)} & T_{s+1}^{(0)} \\ T_{s-1}^{(1)} & T_s^{(0)} \end{vmatrix}, \tag{11}$$

where $$Q_s^{(\lambda)} \equiv K_s\left(g_\lambda, \begin{matrix} f_\lambda \\ h_{\lambda+1}, & i_{\lambda+2}, & h_{\lambda+2}^1, & g_{\lambda+2}^1, & f_{\lambda+2}^1 \end{matrix}\right),$$

$$T_s^{(\lambda)} \equiv K_s\left(h_\lambda, \begin{matrix} g_\lambda, f_\lambda \\ i_{\lambda+1}, & h_{\lambda+1}^1, & g_{\lambda+1}^1, & f_{\lambda+1}^1 \end{matrix}\right).$$

The proof depends on pivotal condensation methods and closely follows that of the lemma. Each determinant in the compound determinant in (11) is replaced by a compound determinant, using identities similar to (6). For example

$$\prod_{\lambda=0}^{s-1} f_\lambda . \; T_s^{(1)} = \begin{vmatrix} Q_s^{(0)} & Q_{s-1}^{(1)} \\ Q_{s+1}^{(0)} & Q_s^{(1)} \end{vmatrix}.$$

We find in this way the relation

$$\prod_{\lambda=-1}^{s-2} f_\lambda . \prod_{\lambda=0}^{s-1} f_\lambda \, K_s\left(\begin{matrix} i_1, & h_1, & g_1, & f_1 \\ & h_1^1, & g_1^1, & f_1^1 \end{matrix}\right)$$

$$= \begin{vmatrix} Q_{s-2}^{(1)} & Q_{s-1}^{(0)} & Q_s^{(-1)} \\ Q_{s-1}^{(1)} & Q_s^{(0)} & Q_{s+1}^{(-1)} \\ Q_s^{(1)} & Q_{s+1}^{(0)} & Q_{s+2}^{(-1)} \end{vmatrix}, \tag{12}$$

it being assumed that $f_\lambda \neq 0$, $\lambda = -1, 0, 1, \ldots, s-1$.

Returning now to (5a) we derive from (12) an expansion for a third order C.F. in terms of *recurrent determinants*, namely

$$F(z_1, z_2, z_3) = \int_0^\infty \frac{d\psi(x)}{(x+z_1)(x+z_2)(x+z_3)}$$

$$= \underset{s \to \infty}{\text{l. i. s.}} \; a_0 \frac{| U_{s-1}, V_s, W_{s+1} |}{| X_{s-1}, V_s, W_{s+1} |} \tag{13}$$

where (i) $$U_s = K_{s-2}\left(\beta_1, \begin{matrix} a_1 \\ \gamma_2, & \delta_3, & \gamma_3, & \beta_3, & a_2 \end{matrix}\right),$$

$$V_s = K_{s-1}\left(\beta_0, \begin{matrix} a_0 \\ \gamma_1, & \delta_2, & \gamma_2, & \beta_2, & a_2 \end{matrix}\right),$$

$$W_s = K_s\left(\beta_{-1}^*, \begin{matrix} a_{-1}^* \\ \gamma_0, & \delta_1, & \gamma_1, & \beta_1, & a_1 \end{matrix}\right),$$

$$X_s = K_{s+1}\left(\beta_{-2}^*, \begin{matrix} a_{-2}^* \\ \gamma_{-1}^*, & \delta_0, & \gamma_0, & \beta_0, & a_0 \end{matrix}\right);$$

(ii) $a_s^* = a_s$, $\beta_s^* = \beta_s$, $\gamma_s^* = \gamma_s$, $s \geqslant 0$;

$a_{-1}^* = a_{-2}^* = 1$, $\beta_{-1}^* = \beta_{-2}^* = 0$, $\gamma_{-1}^* = 0$,

a_s, β_s, γ_s, δ_s being given in (5b);

(iii) the *recurrents* U_s, V_s, W_s, X_s follow

$$y_s = \beta_{s-2}^* y_{s-1} - \gamma_{s-2}^* a_{s-3}^* y_{s-2} + \delta_{s-2} a_{s-3} a_{s-4} y_{s-3} - \gamma_{s-3} a_{s-3} a_{s-4} a_{s-5} y_{s-4} + \beta_{s-4} a_{s-3} a_{s-4} a_{s-5} a_{s-6} y_{s-5} - a_{s-3} a_{s-4} a_{s-5}^2 a_{s-6} a_{s-7} y_{s-6},$$

with $U_2 = 1$, $U_3 = \beta_1$, $U_s = 0$, $s < 2$,

$V_1 = 1$, $V_2 = \beta_0$, $V_s = 0$, $s < 1$,

$W_0 = 1$, $W_1 = 0$, $W_s = 0$, $s < 0$,

$X_{-1} = 1$, $X_0 = 0$, $X_s = 0$, $s < -1$.

It will be seen that each of the elements U_s, V_s, W_s, X_s, occurring in the s^{th} convergent of a third order C.F. follows a sixth order recurrence relation, so that in setting up approximations to $F(z_1, z_2, z_3)$ we have to perform in general twenty-four calculations to obtain each new approximation. Similarly for a C.F. of order n associated with the function $F(z_1, z_2, \ldots, z_n)$, each approximation consists of the ratio of two n^{th} order determinants, an element of either determinant consisting of a *recurrent* which satisfies a recurrence relation of order $2n$. In general then each new approximation to $F(z_1, z_2, \ldots, z_n)$ will involve $2n(n+1)$ calculations, followed by the evaluation of two n^{th} order determinants.[1] We shall consider these more general C.F.'s and the associated recurrence relations in a forthcoming paper.

3. *A Fifth Order Recurrence Relation.*

3.1 We now establish a recurrence relation for the symmetric determinant $K_s(h_1, g_1, f_1)$. Expand K_s by its last row and column.

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[1] A referee has indicated to me that the recurrence relation followed by these two n^{th} order determinants will be of order $\binom{2n}{n}$ in general, or a little less owing to the symmetry involved. Thus for a third order C.F. the numerator and denominator of the s^{th} convergent will very likely satisfy a recurrence relation of order nineteen. Even if this recurrence could be found it might well be too complicated to be of much value, and the method of compound *recurrent* determinants seems to have a distinct advantage for C.F.'s of order three or more.

Then

$$K_s = h_s\, K_{s-1} - g_{s-1}^2\, K_{s-2} + 2g_{s-1} f_{s-2}\, K_{s-2}^* - f_{s-2}^2 h_{s-1}\, K_{s-3} + f_{s-2}^2 f_{s-3}^2\, K_{s-4}, \quad (14)$$

$$s = 4, 5, \ldots,$$

where

$$K_s^* = \begin{vmatrix} & & \\ & \mathbf{K}_{s-1} & \\ & & \\ & & f_{s-2} \\ & & g_{s-1} \\ & & f_{s-1}\ g_s \end{vmatrix},$$

and $\mathbf{K}_{s-1}$ is the matrix consisting of the elements of $K_{s-1}(h_1, g_1, f_1)$. For example,

$$K_2^* = \begin{vmatrix} h_1 & g_1 \\ f_1 & g_2 \end{vmatrix}, \qquad K_3^* = \begin{vmatrix} h_1 & g_1 & f_1 \\ g_1 & h_2 & g_2 \\ & f_2 & g_3 \end{vmatrix}.$$

But expanding K_s^* by its last row, we have

$$K_s^* = g_s\, K_{s-1} - f_{s-1} K_{s-1}^*, \qquad s = 3, 4, \ldots. \quad (15)$$

Eliminating K^* from (14) and (15), we find

$$g_{s-2}\, K_s = \left(h_s g_{s-2} - f_{s-2} g_{s-1}\right) K_{s-1} - \left(g_{s-1} g_{s-2} - h_{s-1} f_{s-2}\right)\left(g_{s-1}\, K_{s-2} - g_{s-2} f_{s-2}\, K_{s-3}\right) - f_{s-3}^2 f_{s-2}\left(h_{s-2} g_{s-1} - f_{s-2} g_{s-2}\right) K_{s-4} + f_{s-2} f_{s-3}^2 f_{s-4}^2 g_{s-1}\, K_{s-5}, \quad (16)$$

$$s = 3, 4, \ldots, \qquad K_{-2} = K_{-1} = 0,\ K_0 = 1.$$

The recurrence relation (16) satisfied by $K_s(h_1, g_1, f_1)$ is of order five. By a slight modification of the method employed here it may be shown that the recurrence relation for the asymmetric determinant $K_s\begin{pmatrix} h_1, & g_1, & f_1 \\ g_1^1, & f_1^1 \end{pmatrix}$ is of order six, but we do not require it in the present context.

There are three interesting special cases:

(a) $f_j = 0$, when (16) reduces to

$$K_s = h_s K_{s-1} - g_{s-1}^2 K_{s-2},$$

as we should expect since $K_s(h_1, g_1, 0)$ is now a "continuant" type of determinant.

(b) $g_j = 0$, when (16) becomes

$$K_s = h_s K_{s-1} - h_{s-1} f_{s-2}^2 K_{s-3} + f_{s-2}^2 f_{s-3}^2 K_{s-4},$$

which is the recurrence relation for the product of two "continuants," and indeed [1]

$$K_{2s}(h_1, 0, f_1) = K_s(h_1^*, f_1^*)\, K_s(h_2^{**}, f_2^{**}),$$

$$K_{2s+1}(h_1, 0, f_1) = K_{s+1}(h_1^*, f_1^*)\, K_s(h_2^{**}, f_2^{**}),$$

where $h_s^* = h_{2s-1}$, $f_s^* = f_{2s-1}$,

$h_s^{**} = h_{2s}$, $f_s^{**} = f_{2s}$.

We shall treat the third example of reducibility in § 4, for it turns out to have several applications.

3.2 Now applying (16) to (4a) we may write the second order C.F. as

$$F(z_1, z_2) = \underset{s\to\infty}{\text{l.i.s.}}\ a_0 \frac{u_s}{v_s}, \tag{17}$$

where $K_s(\gamma_0, \beta_0, \alpha_s) = v_s \prod_{\lambda=1}^{s} a_\lambda$, $K_{s-1}(\gamma_1, \beta_1, \alpha_1) = u_s \prod_{\lambda=1}^{s} a_\lambda$,

and u_s, v_s follow the recurrence

$$\begin{aligned} a_s \beta_{s-3}^1 y_s = {} & (\gamma_{s-1}\beta_{s-3}^1 - a_{s-1}\beta_{s-2}^1) y_{s-1} - (\beta_{s-2}^1 \beta_{s-3}^1 - \gamma_{s-2})(\beta_{s-2}^1 y_{s-2} - \beta_{s-3}^1 y_{s-3}) \\ & - (\gamma_{s-3}\beta_{s-2}^1 - a_{s-2}\beta_{s-3}^1) y_{s-4} + a_{s-3}\beta_{s-2}^1 y_{s-5}, \end{aligned} \tag{18}$$

where $\beta_s^1 \surd a_{s+1} = \beta_s$, and α_s, β_s, γ_s, a_s are given in (4b), the initial values being

$$u_0 = 0, \quad a_1 u_1 = 1, \quad a_1 a_2 u_2 = \gamma_1, \quad u_s = 0,\ s < 0;$$

$$v_0 = 1, \quad a_1 v_1 = \gamma_0, \quad a_1 a_2 v_2 = \gamma_0\gamma_1 - \beta_0^2, \quad v_s = 0,\ s < 0.$$

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[1] The result has been noted by T. Muir, *Proc. Edinburgh Math. Soc.* ii (1884), 16-18.

3.3 As an illustration consider the "J" fraction expansion, convergent for $z > 0$.

$$\int_0^1 \frac{dx}{x+z} = \cfrac{1}{z+\frac{1}{2}-\cfrac{a_1}{z+\frac{1}{2}-\cfrac{a_2}{z+\frac{1}{2}-\cfrac{a_3}{z+\frac{1}{2}-\ldots}}}}, \tag{19}$$

where $a_s = s^2/(16s^2-4)$, from which we deduce the second order C.F. expansion

$$\int_0^1 \frac{dx}{x^2+z^2} = z^{-1} \arctan z^{-1} = \underset{s\to\infty}{\text{l.i.s.}} \frac{u_s}{v_s}, \quad z \neq 0, \tag{20}$$

where u_s and v_s follow

$$\begin{aligned} a_s y_s = {} & (z^2+a_s+\tfrac{1}{4})\, y_{s-1} + (z^2+a_{s-1}+a_{s-2}-\tfrac{3}{4})\,(y_{s-2}-y_{s-3}) \\ & - (z^2+a_{s-3}+\tfrac{1}{4})\, y_{s-4} + a_{s-3}\, y_{s-5}, \qquad s = 3,\ 4,\ \ldots, \end{aligned} \tag{21}$$

and the initial values are

$$\begin{aligned} & u_s = 0,\ s<0, \qquad u_0 = 0,\ u_1 = 12,\ u_2 = 3\,(60z^2+24); \\ & v_s = 0,\ s<0, \qquad v_0 = 1,\ v_1 = 4\,(3z^2+1),\ v_2 = 3\,(60z^2+44z+3). \end{aligned} \tag{21a}$$

For example, using (21) it will be found that

$$\begin{cases} 3u_3 = 16\,(525z^4+410z^2+45), \\ 3v_3 = 16\,(525z^6+585z^4+135z^2+3), \end{cases}$$

$$\begin{cases} 3u_4 = 132{,}300z^6+153{,}300z^4+41{,}300z^2+1{,}800, \\ 3v_4 = 132{,}300z^8+197{,}400z^6+80{,}640z^4+8{,}100z^2+75. \end{cases}$$

The expansion indicated in (20)-(21) is not the same (apart from the approximations for $s = 0, 1$) as the even part of the *hypergeometric* C.F.

$$z^{-1} \arctan z^{-1} = \cfrac{1}{z^2+\cfrac{b_1}{1+z^2+\cfrac{b_2}{1+z^2+\cfrac{b_3}{1+z^2+\ldots}}}},$$

where $b_s = s^2/(4s^2-1)$.

4. *A Reducible Case of the Fifth Order Recurrence Relation.*

4.1 The recurrence relation (16) reduces to a fourth order one when the C.F. in (2) takes on the special form

$$F(z) = \cfrac{a_0}{z-\cfrac{a_1}{z-\cfrac{a_2}{z-\cfrac{a_3}{z-\ldots}}}}. \tag{22}$$

Proceeding formally at first and writing $K_{s-1}(\gamma_1, \beta_1, \alpha_1) = T_s^*$, $K_s(\gamma_0, \beta_0, \alpha_0) = T_s$, we find that the recurrence for T_s^* and T_s becomes

$$y_s = (q + a_s)\, y_{s-1} + a_{s-1}\,(q - p^2 + a_{s-1} + a_{s-2})\, y_{s-2}$$
$$- a_{s-1}\, a_{s-2}\,(q - p^2 + a_{s-1} + a_{s-2})\, y_{s-3} - a_{s-1}\, a_{s-2}\, a_{s-3}\,(q + a_{s-3})\, y_{s-4}$$
$$+ a_{s-1}\, a_{s-2}\, a^2_{s-3}\, a_{s-4}\, y_{s-5}, \qquad s = 3, 4, \ldots, \tag{23}$$

where $q = z_1 z_2$, $p = z_1 + z_2$, $\quad T_s = 0,\ s < 0, \quad T^*_s = 0,\ s \leqslant 0.$

Now (23) may be written

$$\Phi_s(y) - a_{s-1}\, \Phi_{s-1}(y) = 0, \tag{24}$$

where
$$\Phi_s(y) \equiv y_s - (q + a_s - a_{s-1})\, y_{s-1} - a_{s-1}(2a_{s-1} - p^2 + 2q)\, y_{s-2} \tag{25}$$
$$- a_{s-1}\, a_{s-2}\,(q + a_{s-2} - a_{s-1})\, y_{s-3} + a_{s-1}\, a^2_{s-2}\, a_{s-3}\, y_{s-4}.$$

But it is easily verified that $\Phi_2(T) = 0$. Hence from (25)

$$T_s = (q + a_s - a_{s-1})\, T_{s-1} + a_{s-1}\,(2a_{s-1} - p^2 + 2q)\, T_{s-2} \tag{26}$$
$$+ a_{s-1}\, a_{s-2}\,(q + a_{s-2} - a_{s-1})\, T_{s-3} - a_{s-1}\, a^2_{s-2}\, a_{s-3}\, T_{s-4}, \qquad s = 2, 3, \ldots,$$

with $\quad T_0 = 1, \quad T_1 = q + a_1, \quad T_s = 0,\ s < 0.$

Similarly it will be found that

$$\Psi_s(T^*) - a_{s-1}\, \Psi_{s-1}(T^*) = 0, \tag{27}$$

where $\quad \Psi_s(T^*) \equiv \Phi_s(T^*) - 2 \prod_{\lambda=0}^{s-1} a_\lambda.$

But since $\Psi_2(T^*) = 0$ it follows that the recurrence for T^*_s is

$$T^*_s = (q + a_s - a_{s-1}) T^*_{s-1} + a_{s-1}\,(2a_{s-1} - p^2 + 2q)\, T^*_{s-2}$$
$$+ a_{s-1} a_{s-2}\,(q + a_{s-2} - a_{s-1})\, T^*_{s-3} - a_{s-1} a^2_{s-2} a_{s-3}\, T^*_{s-4} + 2 \prod_{\lambda=0}^{s-1} a_\lambda,$$
$$s = 2, 3, \ldots, \tag{28}$$

with $\quad T^*_1 = a_0, \quad T^*_s = 0,\ s \leqslant 0.$

4.2 Returning to §2 we observe that in (1) $b_s > 0$, $s = 1, 2, \ldots$. Hence c_s cannot be zero for a Stieltjes C.F. We may ask the question then as to how the value of a_s in (22) must be restricted so that for certain values of z_1 and z_2 it will be true to assert that

$$\frac{F(z_1) - F(z_2)}{z_2 - z_1} = \lim_{s \to \infty} \frac{T^*_s}{T_s}.$$

We can give at this stage only a partial answer to this question. First we refer to the theory of the *Hamburger moment problem*. Let the expansion in descending powers of z of $F(z)$ be

$$F(z) \sim \frac{\mu_0}{z} + \frac{\mu_2}{z^3} + \frac{\mu_4}{z^5} + \dots \tag{29}$$

and assume that

$$\begin{cases} a_s \geqslant 0, & s = 0, 1, 2, \dots, \\ \sum_{n=1}^{\infty} \mu_{2n}^{-1/2n} = \infty . \end{cases} \tag{30}$$

Then there exists a unique bounded non-decreasing function $\psi(x)$ in the interval $(-\infty, \infty)$ such that[1] $\int_{-\infty}^{\infty} x^{2n} d\psi(x) = \mu_{2n}$, $\int_{-\infty}^{\infty} x^{2n+1} d\psi(x) = 0$.

All we have to do now is to justify the Parseval expansion for $\int_{-\infty}^{\infty} \{f(x)\}^2 d\bar{\psi}(x)$ where $f(x) = \{(x+z_1)\ (x+z_2)\}^{-1}$,

$$\bar{\psi}(x) = \int_{-\infty}^{x} (z_1+t)\,(z_2+t)\, d\psi(t),$$

the argument being similar to that used in §2 of **S4**. It turns out then, using the theorem of M. Riesz (Shohat and Tamarkin, *loc. cit.*, p. 62) regarding the solution of a *determined* Hamburger moment problem, that under the conditions in (30) we have

$$F(z_1, z_2) = \int_{-\infty}^{\infty} \frac{d\psi(x)}{(x+z_1)\,(x+z_2)} = \text{l. i. s.}_{s\to\infty} \frac{T_s^*}{T_s}, \quad \begin{cases} z_1 = x_1 + iy_1, \\ z_2 = x_1 - iy_1, \end{cases} \tag{31}$$

provided in addition $(x+z_1)\,(x+z_2) > 0$ for all real x. Again using (34) and (35) of **S4** we can set up a decreasing sequence of upper bounds, the difference between corresponding s^{th} approximants being $\prod_{\lambda=0}^{s} a_\lambda / (y_1^2 T_s)$, where $y_1 = \text{Im} z$.

Secondly it may be possible to justify (31) if we are given a definite integral for the C.F. in (22) of the form $\int_a^b \frac{d\psi(x)}{x+z}$ and can justify the application of Parseval's theorem as in §B of **S2**. It would

[1] See for example Shohat, J. A. and Tamarkin, J.D., *The Problem of Moments*, p. 5 and p. 19 (New York : American Mathematical Society, 1943).

be of some interest to know what are the weakest restrictions on the a's in (22) to justify a statement similar to (31).

4.2. We now give several examples in illustration.

Example 1. Let

$$F(z) = \tan(\tfrac{1}{2}z^{-1}),$$

so that $$\mu_{2n} = 2B_{n+1}(2^{2n+2}-1)/(2n+2)!,$$

where B_n is a *Bernoulli* number and $B_1 = 1/6$, $B_2 = 1/30$ etc. Using Lambert's C.F.[1] for $F(z)$ we have $a_0 = \frac{1}{2}$, $a_s = (16s^2-4)^{-1}$, $s = 1, 2, \ldots$ Moreover it is easily verified that $\mu_{2n}^{1/2n} \sim 1/\pi$, so that (30) is satisfied. Hence with the appropriate value of a_s in (26) and (28) we have

$$\frac{\sinh y_1^1}{y_1(\cosh y_1^1 + \cos x_1^1)} = \text{l. i. s.} \frac{T_s^*}{T_s}, \quad \begin{cases} y_1 \neq 0, \\ z_1 = x_1 + iy_1 \\ z_2 = x_1 - iy_1. \end{cases} \tag{32}$$

where $x_1^1(x_1^2 + y_1^2) = x_1$, $y_1^1(x_1^2 + y_1^2) = y_1$. In particular if $x_1 = 3$, $y_1 = 4$ then $\lambda T_4^* = 398,499,385,800$, $\lambda T_4 = 19,896,118,681,110$ where $\lambda^{-1} = a_4 a_3 a_2 a_1 a_0$, giving the lower bound $0{\cdot}020,029,001,243,260$, and using the corresponding upper bound it turns out that the error in this cannot exceed $1{\cdot}6 \times 10^{-15}$.

In a similar way, taking $F(z)$ to be $2z - \cot(\frac{1}{2}z^{-1})$, so that $a_0 = 1/6$, $a_s = [(4s+2)(4s+6)]^{-1}$, $s = 1, 2, \ldots$, it may be shown that

$$\frac{\sinh y_1^1}{y_1(\cosh y_1^1 - \cos x_1^1)} - 2 = \text{l. i. s.} \frac{T_s^*}{T_s}, \quad y_1 \neq 0. \tag{33}$$

In each case the recurrence relation for T_s is (26) and that for T_s^* is (28) with the appropriate value of a_s, $s = 0, 1, 2, \ldots$.

Example 2. It has been indicated by Stieltjes[2] that

$$\tfrac{1}{2}\int_{-\infty}^{\infty} \frac{\operatorname{sech}(\frac{1}{2}\pi x)\,dx}{x+z} = \cfrac{1}{z - \cfrac{1^2}{z - \cfrac{2^2}{z - \cfrac{3^2}{z - \ldots}}}}, \quad \operatorname{Im}(z) \neq 0, \tag{34a}$$

$$\tfrac{1}{2}\int_{-\infty}^{\infty} \frac{x \operatorname{cosech}(\frac{1}{2}\pi x)\,dx}{x+z} = \cfrac{1}{z - \cfrac{1.2}{z - \cfrac{2.3}{z - \cfrac{3.4}{z - \ldots}}}}, \quad \operatorname{Im}(z) \neq 0. \tag{34b}$$

It may be shown that the Hamburger moment problem is determined in each case[3], and it follows that for $z_1 = x_1 + iy_1$, $z_2 = x_1 - iy$, $y_1 \neq 0$

[1] See for example Perron, O., *Die Lehre von den Kettenbrüchen*, p. 354, (Berlin, 1913).

[2] *Correspondance d'Hermite et de Stieltjes*, p. 360 (Paris, 1905).

[3] Compare also Wall, H. S., *Continued Fractions*, p. 366, Example 2 (New York, 1948). We may also recall that the Hamburger moment problem

$$\mu_n = \int_{-\infty}^{\infty} x^n e^{-by}\,dx, \quad y = |x|^a, \; a > 1, \; b > 0, \; n = 0, 1, 2, \ldots \text{ is determined.}$$

the second order C.F.'s indicated in (31) converge to the corresponding value of $F(z_1, z_2)$.

Example 3. Let $F(z) = \int_{-\infty}^{\infty} \frac{g(x)dx}{x+z}$,

where $g(x) = e^{-\frac{1}{2}x^2}/\sqrt{(2\pi)}$, with Im $(z) \neq 0$, so that

$$F(z) = \frac{1}{z} - \frac{1}{z} - \frac{2}{z} - \frac{3}{z} - \ldots . \tag{35}$$

From P. 3, §B of **S2** we can conclude that the second order C.F. converges for $y_1 \neq 0$, and indeed

$$\int_{-\infty}^{\infty} \frac{g(x)dx}{(x+x_1)^2 + y_1^2} = \text{l. i. s.}_{s\to\infty} \frac{T_s^*}{T_s},$$

where

$$T_s^* = (x_1^2 + y_1^2 + 1)\ T_{s-1}^* + 2\ (s-1)\ (s - 1 + y_1^2 - x_1^2)\ T_{s-2}^*$$
$$+ (s-1)\ (s-2)\ (x_1^2 + y_1^2 - 1)\ T_{s-3}^* - (s-1)\ (s-2)^2\ (s-3)\ T_{s-4}^*$$
$$+ 2(s-1)!, \qquad s = 2, 3, \ldots, \tag{36}$$

and the recurrence for T_s is exactly the same except that the factorial term is omitted. The initial values are

$$T_s^* = 0,\ s < 1, \qquad T_1^* = 1$$
$$T_s = 0,\ s < 0, \qquad T_0 = 1, \qquad T_1 = x_1^2 + y_1^2 + 1.$$

A numerical example will be found in §C 4 of **S2**.

5. *A Recurrence Relation with* Even *and* Odd *Parts.*

5.1 For the C.F. given in (1) there are recurrence relations corresponding to the *even* and *odd* parts, namely

$$w_{2s}(z) = w_{2s-1}(z) + b_{2s}\, w_{2s-2}(z),$$
$$w_{2s+1}(z) = z w_{2s}(z) + b_{2s+1}\, w_{2s-1}(z). \tag{37}$$

The question naturally arises as to whether there is a similar structure for higher order C.F.'s. We give here the result for a second order C.F. Starting with the form of the denominator of (4a) given in (30) of **S4**, we have, assuming for the moment $z_1 \neq z_2$,

$$K_s(\gamma_0, \beta_0, \alpha_0) = |\ w_{2s}(z_1),\ w_{2s+2}(z_2)\ | \div (z_2 - z_1), \tag{38}$$
$$= k_{2s},$$

say. Using (37) we readily find that

$$k_{2s} = y_{2s} + b_{2s+1}\, b_{2s}\, k_{2s-2}, \tag{39}$$

where $y_s = w_s(z_1)w_s(z_2)$.

Similarly if

$$k_{2s-1} = \mid w_{2s-1}(z_1),\ w_{2s+1}(z_2) \mid \div (z_2 - z_1), \tag{40}$$

then

$$k_{2s-1} = y_{2s-1} + b_{2s}\, b_{2s-1} k_{2s-3}. \tag{41}$$

But

$$y_{2s} = y_{2s-1} + b_{2s}^2\, y_{2s-2} + b_{2s}\, \Theta_{2s-1}, \tag{42}$$

where

$$\Theta_{2s-1} = w_{2s-1}(z_1)\, w_{2s-2}(z_2) + w_{2s-1}(z_2)\, w_{2s-2}(z_1)$$
$$= (z_1 + z_2)\, y_{2s-2} + b_{2s-1}\,[w_{2s-2}(z_1)w_{2s-3}(z_2) + w_{2s-2}(z_2)w_{2s-3}(z_1)].$$

Hence $\Theta_{2s-1} = (z_1 + z_2)y_{2s-2} + 2b_{2s-1}y_{2s-3} + b_{2s-1}b_{2s-2}\Theta_{2s-3}$,

and so from (42) we have

$$y_{2s} - y_{2s-1} - b_{2s}^2 y_{2s-2} = b_{2s}(z_1+z_2)y_{2s-2} + 2b_{2s}b_{2s-1}y_{2s-3}$$
$$+ b_{2s}b_{2s-1}(y_{2s-2} - y_{2s-3} - b_{2s-2}^2 y_{2s-4}),$$

from which, using (39) and (41), we deduce

$$k_{2s} = k_{2s-1} + b_{2s}(z_1 + z_2 + b_{2s+1} + b_{2s} + b_{2s-1})k_{2s-2}$$
$$- b_{2s}b_{2s-1}b_{2s-2}(z_1 + z_2 + b_{2s} + b_{2s-1} + b_{2s-2})k_{2s-4}$$
$$- b_{2s}b_{2s-1}b^2{}_{s-2}\, b_{2s-3}k_{2s-5} + b_{2s}b_{2s-1}b_{2s-2}^2 b_{2s-3}b_{2s-4}k_{2s-6}. \tag{43}$$

5.2 Similarly from (37) it follows that

$$y_{2s+1} = z_1z_2y_{2s} + b_{2s+1}^2 y_{2s-1} + b_{2s+1}\Phi_{2s}, \tag{44}$$

where

$$\Phi_{2s} = z_1w_{2s}(z_1)w_{2s-1}(z_2) + z_2w_{2s}(z_2)\, w_{2s-1}(z_1)$$
$$= (z_1+z_2)y_{2s-1} + b_{2s}[z_1w_{2s-2}(z_1)w_{2s-1}(z_2) + z_2w_{2s-2}(z_2)w_{2s-1}(z_1)],$$

or

$$\Phi_{2s} = (z_1 + z_2)y_{2s-1} + 2z_1z_2b_{2s}y_{2s-2} + b_{2s}b_{2s-1}\Phi_{2s-2}. \tag{45}$$

Thus

$$y_{2s+1} = z_1z_2y_{2s} + b_{2s+1}^2 y_{2s-1} + b_{2s+1}(z_1 + z_2)y_{2s-1} + 2z_1z_2b_{2s+1}b_{2s}y_{2s-2}$$
$$+ b_{2s+1}b_{2s}(y_{2s-1} - z_1z_2y_{2s-2} - b_{2s-1}^2 y_{2s-3}), \tag{46}$$

and so by (39) and (41) it appears that

$$k_{2s+1} = z_1z_2k_{2s} + b_{2s+1}(z_1 + z_2 + b_{2s+2} + b_{2s+1} + b_{2s})k_{2s-1}$$
$$- b_{2s+1}b_{2s}b_{2s-1}(z_1 + z_2 + b_{2s+1} + b_{2s} + b_{2s-1})k_{2s-3}$$
$$- b_{2s+1}b_{2s}b_{2s-1}b_{2s-2}z_2z_1k_{2s-4} + b_{2s+1}b_{2s}b_{2s-1}^2 b_{2s-2}b_{2s-3}k_{2s-5}. \tag{47}$$

In the derivation of the recurrence formulæ (43) and (47) it has been assumed that s is large enough to avoid initial value idiosyncrasies. Allowance being made for these we finally have the theorem :

If the Stieltjes moment problem is determined and

$$F(z) = \int_0^\infty \frac{d\psi(x)}{x+z} = \cfrac{b_1}{z+1+\cfrac{b_2}{1+\cfrac{b_3}{z+1+\cfrac{b_4}{1+\cfrac{b_5}{z+1+\cfrac{b_6}{1+\ldots}}}}}},$$

then

$$\text{l. i. s.}\frac{k^*_{2s}}{k_{2s}} = \lim_{s\to\infty}\frac{k^*_s}{k_s} = \frac{F(z_2)-F(z_1)}{z_1-z_2}, \tag{48}$$

where

(i) k^*_s *and* k_s *follow, for* $s = 2, 3, \ldots,$

$$w_{2s-1} = z_1z_2w_{2s-2} + \alpha_{2s-1}w_{2s-3} - \beta_{2s-1}w_{2s-5} - z_1z_2\gamma_{2s-1}w_{2s-6} + \delta_{2s-1}w_{2s-7}$$

$$w_{2s} = w_{2s-1} + \alpha_{2s}w_{2s-2} - \beta_{2s}w_{2s-4} - \gamma_{2s}w_{2s-5} + \delta_{2s}w_{2s-6},$$

(ii) $k^*_0 = 0,\ k^*_1 = k^*_2 = b_1, \qquad k^*_s = 0,\ s<0,$

$$k_0 = 1,\ k_1 = z_1z_2,\ k_2 = z_1z_2 + b_2(z_1+z_2+b_3+b_2),\ k_s = 0,\ s<0,$$

(iii) $\alpha_s = b_s(z_1+z_2+b_{s+1}+b_s+b_{s-1}),\ \beta_s = b_sb_{s-2}a_{s-1},$

$$\gamma_s = b_sb_{s-1}b_{s-2}b_{s-3}, \qquad \delta_s = b_sb_{s-1}b^2_{s-2}b_{s-3}b_{s-4},$$

(iv) $(x+z_1)(x+z_2) > 0$ for[1] $x \geqslant 0.$

It may be established that the 'odd' part k^*_{2s-1}/k_{2s-1} arises from the second order C.F. associated with the integral in the expression

$$z_1z_2\left(\frac{F(z_2)-Fz_1)}{z_1-z_2}\right) = b_1 - \int_0^\infty \frac{x(x+z_1+z_2)d\psi(x)}{(x+z_1)(x+z_2)}$$

where $xd\psi(x)$ is taken as the weight function. The 'odd' part of the sequence, unlike that of a Stieltjes C.F., does not in general provide a set of decreasing upper bounds, but there is a remarkable property which we now consider.

5.2 We shall prove the following identities :

$$t_{2s+1} - t_{2s} = \frac{B_{2s+1}\,U^{(1)}_{2s+1}\,U^{(1)}_{2s+2}}{k_{2s}\,k_{2s+1}},\ s = 0, 1, 2, \ldots \tag{49a}$$

$$t_{2s} - t_{2s-1} = -\frac{B_{2s}U^{(1)}_{2s}U^{(1)}_{2s+1}}{k_{2s-1}\,k_{2s}},\ s = 1, 2, \ldots \tag{49b}$$

where

$$U^{(1)}_s = \frac{w_s(z_2) - w_s(z_1)}{z_2 - z_1}, \qquad B_s = \prod_{\lambda=1}^{s} b_\lambda, \qquad t_s = k^*_s/k_s.$$

---

[1] It has been assumed throughout that $z_1 \neq z_2$, but it is easily shown that the theorem still holds if $z_1 = z_2$ and $(x+z_1)^2 > 0$ for $x \geqslant 0$.

Introducing the expressions appearing in (30) and (47) of §4 for t_s, we have ($z_1 \neq z_2$)

$$b_{2s-2}(z_1 - z_2)^3 k_{2s} k_{2s-1}(t_{2s+1} - t_{2s}) \tag{50}$$

$$= | w_{2s+1}(z_1),\ w_{2s+2}(z_2) | X_s - \{z_1 w_{2s-1}(z_2) w_{2s+2}(z_1) - z_2 w_{2s+1}(z_1) w_{2s+2}(z_2)\} Y_s,$$

where

$$X_s = z_1 \xi_{2s+1}(z_1, z_2) + z_2 \xi_{2s+1}(z_2, z_1) - (z_1 + z_2) B_{2s+2},$$

$$Y_s = 2B_{2s+2} - \eta_{2s+1}(z_1, z_2) - \eta_{2s-1}(z_2, z_1),$$

$$\xi_{2s+1}(z_1, z_2) = \chi_{2s+1}(z_2) w_{2s+2}(z_1) - \chi_{2s+2}(z_1) w_{2s+1}(z_2),$$

$$\eta_{2s+1}(z_1, z_2) = \chi_{2s+1}(z_1) w_{2s+2}(z_2) - \chi_{2s+2}(z_1) w_{2s+1}(z_2).$$

Now since $\eta_{2s+1}(z, z) = B_{2s+2}$, the right member of (50) becomes

$$(z_2 - z_1)\{w_{2s+1}(z_1) w_{2s+2}(z_2) \mu_{2s+1}(z_1, z_2) + w_{2s+1}(z_2) w_{2s+2}(z_1) \mu_{2s+1}(z_2, z_1)\},$$

where $$\mu_{2s+1}(z_1, z_2) = \chi_{2s+2}(z_1) w_{2s+1}(z_2) - \chi_{2s+1}(z_2) w_{2s+2}(z_1) + B_{2s+2},$$

and from which (49a) follows after simplification.[1] The proof of (49b) is similar. We now deduce the expansion, valid under the conditions of the theorem in (48),

$$F(z_1, z_2) = \sum_{s=1}^{\infty} \frac{(-1)^{s+1} B_s U_s^{(1)} U_{s+1}^{(1)}}{k_{s-1} k_s}. \tag{51}$$

This is the third type of expansion for $F(z_1, z_2)$, the other two, given in earlier parts, being

$$F(z_1, z_2) = \sum_{s=0}^{\infty} \frac{B_{2s+1}\{U_{2s+2}^{(1)}\}^2}{k_{2s} k_{2s+2}}, \tag{52}$$

$$= \frac{b_1}{z_1 z_2} - \sum_{s=1}^{\infty} \frac{B_{2s} U_{2s+1}^{(0)} U_{2s+1}^{(1)}}{k_{2s-1} k_{2s+1}}, \tag{53}$$

where $$U_s^{(0)} = \begin{vmatrix} z_1 & z_2 \\ w_s(z_1) & w_s(z_2) \end{vmatrix} \div (z_2 - z_1).$$

The expansions in (51)-(53) bear a striking resemblance to those for a first order C.F., namely

$$F(z) = \sum_{s=1}^{\infty} \frac{(-1)^{s+1} B_s}{w_{s-1} w_s} \tag{54}$$

$$= \sum_{s=0}^{\infty} \frac{B_{2s+1}}{w_{2s} w_{2s+2}} \tag{55}$$

$$= \frac{b_1}{z} - z \sum_{s=1}^{\infty} \frac{B_{2s}}{w_{2s-1} w_{2s+1}} \tag{56}$$

where $w_s \equiv w_s(z)$.

[1] The results in (49) still hold if $z_1 = z_2$, and we merely introduce the *confluent* forms of k_s and $U_s^{(1)}$.

Under the conditions set down in §2, (55) gives an increasing sequence (56) a decreasing sequence and (54) an enveloping sequence[1], provided z is real and positive. Correspondingly (52) gives an increasing sequence and (53) a converging sequence when z_1 and z_2 are complex conjugates with $\mathrm{Im}\, z_1 \neq 0$. We can go a little further than this. Suppose the interval $(0, \infty)$ is "reducible," i.e., it may reduce to a sub-interval (a, b) if $\psi(x)=\psi(a)$ for $x<a$, $a \geqslant 0$, and $\psi(x)=\psi(b)$ for $x>b$, $b>a$. Now using the fact that the zeros of $w_s(x)$ are distinct and lie entirely within[2] $(-b, -a)$ and recalling that the degrees of the highest terms in $w_{2s}(x)$ and $w_{2s+1}(x)$ are s and $s+1$ respectively, we easily see that[3]

$$\begin{cases} U^{(1)}_{4s+1} > 0, \; U^{(1)}_{4s+2} > 0, \text{ for } z_1 \leqslant z_2 < -b, \\ U^{(1)}_{4s} < 0, \quad U^{(1)}_{4s+3} < 0, \end{cases} \tag{57a}$$

$$U^{(1)}_{4s} > 0, \text{ for } z_2 \geqslant z_1 > -a. \tag{57b}$$

Hence for $z_1 \leqslant z_2 < -b$ it follows from (49) that $\{t_s\}$ is an increasing sequence, whereas this is not necessarily so for $z_2 \geqslant z_1 > -a$. Again suppose $z_2 > 0$, $z_1 < -b$ but $z_2 \geqslant |z_1|$. Then $U^{(1)}_s$ is clearly positive. But it is known (see **S1**, (18)) that

$$\int_a^b (x+z_1)(x+z_2)\, q_s^2(x)\, d\psi(x) = B_{2s+1}(z_2 - z_1)^2 k_{2s} k_{2s+2}, \tag{58}$$

where $\{q_s(x)\}$ is an orthogonal system with respect to the weight function $(x+z_1)(x+z_2)\, d\psi(x)$. We deduce from (58), since $k_0 = 1$, the inequalities for the even denominators

$$\begin{cases} k_{4s} > 0, & k_{4s+2} < 0, \\ k_{4s+1} < 0, & k_{4s+3} > 0. \end{cases} \qquad \begin{cases} z_2 > 0 \\ z_1 < -b. \end{cases} \tag{59}$$

The result given for the odd denominators may be proved in a similar way. Hence $\{t_s\}$ is a decreasing sequence for $z_2 > 0$, $z_1 < -b$, $z_2 \geqslant |z_1|$. Lastly suppose z_1 and $z_2 > 0$. Then $U^{(1)} > 0$, $U^{(0)}_s > 0$, and so from (52) and (53) $\{t_s\}$ is an enveloping sequence. A summary of the various possibilities appears in Table 1.

---

[1] i.e. $s_{2r} < F < s_{2r+1}$ where s_r is the sum of the first r terms of the series.

[2] Exceptionally, $w_{2s+1}(x)$ always has a zero $x = 0$.

[3] It is assumed now that z_1, z_2 are entirely real.

TABLE 1.

NATURE OF SERIES FOR SECOND ORDER C.F.

Arguments	Series (51)	(52)	(53)
$z_1 = \bar{z}_2$, $\mathrm{Im}(z_1) \neq 0$.	C	I	C
$z_1 < -b$, $z_2 < -b$.	I	I	I
$z_2 > 0$, $z_1 < -b$, $z_2 \geqslant \mid z_1 \mid$.	D	D	D
$z_1 > 0$, $z_2 > 0$.	E	I	D

C = Converges, D = Decreases, I = Increases, E = Envelopes.

6. *Numerical Illustrations.*

Example 1. $$F(z) = \cfrac{1}{z+\cfrac{1}{1+\cfrac{1}{z+\cfrac{1}{1+\ldots}}}} = \frac{1}{2\pi}\int_0^4 \frac{\sqrt{(4x^{-1}-1)}\,dx}{x+z},$$

for $z > 0$.

The second order C.F. for $F(z, z) = \{z\sqrt{(z^2+4z)}\}^{-1}$, $z > 0$, is given by $\lim_{s\to\infty} t_s$ where

$$k_1^* = 1,\ k_2^* = 1,\ k_1 = z^2,\ k_2 = z^2 + 2z + 2,$$

and k_s^*, k_s follow

$$w_{2s-1} = z^2 w_{2s-2} + (2z+3)w_{2s-3} - (2z+3)w_{2s-5} - z^2 w_{2s-6} + w_{2s-7},$$
$$w_{2s} = w_{2s-1} + (2z+3)w_{2s-2} - (2z+3)w_{2s-4} - w_{2s-5} + w_{2s-6}, \qquad s = 2, 3, \ldots,$$
$$k_s^* = 0,\ s \leqslant 0, \qquad k_s = 0,\ s < 0.$$

From Table 1 the sequence $\{t_s\}$ is enveloping. In particular with $z=1$ the limit of the convergents is $1/\sqrt{5}$ and the first twenty are shown in Table 2

TABLE 2.

s	k_s^*	k_s	t_s	s	k_s^*	k_s	t_s
1	1	1	1·0	11	10866	24276	0·4476
2	1	5	0·2	12	28416	63565	0·4470
3	6	10	0·6	13	74431	166405	0·447288
4	11	30	0·37	14	194821	435665	0·447181
5	36	74	0·49	15	510096	1140574	0·447227
6	85	199	0·43	16	1335395	2986074	0·447208
7	235	515	0·456	17	3496170	7817630	0·447216
8	600	1355	0·443	18	9153025	20466835	0·4472125
9	1590	3540	0·449	19	23963005	53582855	0·4472140
10	4140	9276	0·4463	20	62735880	140281751	0·4472134
				∞	———	———	0·4472136

Example 2. $$F(z) = \cfrac{1}{z+}\ \cfrac{\lambda}{1+}\ \cfrac{\lambda}{z+}\ \cfrac{\lambda}{1+}\ \ldots,$$

$$= \frac{1}{2\pi\lambda}\int_0^{4\lambda} \frac{\sqrt{\{x(4\lambda - x)\}}dx}{x(x+z)}, \qquad \begin{cases}\lambda > 0\\ z > 0.\end{cases}$$

With $\lambda = \frac{1}{4}$ we consider $F(\sqrt{2}, -\sqrt{2}) = \lim_{s\to\infty} t_s$,

where $k_s^* = X_s/4^s$, $k_s = Y_s/4^s$, and the recurrence for X_s and Y_s is

$$w_{2s-1} = -8w_{2s-2} + 3w_{2s-3} - 3w_{2s-5} + 8w_{2s-6} + w_{2s-7},$$

$$w_{2s} = 4w_{2s-1} + 3w_{2s-2} - 3w_{2s-4} - 4w_{2s-5} + w_{2s-6}, \qquad s = 2, 3, \ldots,$$

with

$$X_1 = 4,\ X_2 = 16,\ Y_0 = 1,\ Y_1 = -8,\ Y_2 = -30,$$

$$X_s = 0,\ s \leqslant 0, \qquad Y_s = 0,\ s < 0.$$

The sequence $\{-t_s\}$ steadily increases to $\frac{1}{2}\{\sqrt{(2+\sqrt{2})} - \sqrt{(2-\sqrt{2})}\}$, and a few values are stated in Table 3.

TABLE 3.

s	$-t_s$	s	$-t_s$
1	0·5	8	0·541186
2	0·53	9	0·541193
3	0·537	10	0·541195
4	0·5396	11	0·5411958
5	0·5408	12	0·54119602
6	0·5411	13	0·54119608
7	0·54116	∞	0·54119610

Example 3. From

$$F(z) = \cfrac{1}{z+}\ \cfrac{a}{1+}\ \cfrac{1}{z+}\ \cfrac{a+1}{1}\ \cfrac{2}{+z+}\ \cfrac{a+2}{1}\ \cfrac{3}{+z+}\ \cfrac{a+3}{1}\ +\ldots$$

$$= \frac{1}{\Gamma(a)}\int_0^\infty \frac{e^{-x}x^{a-1}dx}{x+z}, \qquad a > 0,\ z > 0,$$

we derive the C.F. for $F(iz, -iz)$ which we write

$$\Phi(a, z) = \frac{1}{\Gamma(a)}\int_0^\infty \frac{e^{-x}x^{a-1}\,dx}{x^2+z^2} = \underset{s\to\infty}{\text{l. i. s.}}\,(t_{2s}), \qquad z \neq 0,$$

where k_s^* and k_s follow

$$w_{2s-1} = z^2 w_{2s-2} + (s-1)(2a+3s-4)w_{2s-3} - (s-1)(s-2)(a+s-2)(a+3s-5)w_{2s-5} - (s-1)(s-2)(a+s-2)(a+s-3)z^2 w_{2s-6} + (s-1)(s-2)^2(s-3)(a+s-2)(a+s-3)w_{2s-7},$$

$$w_{2s} = w_{2s-1} + (a+s-1)(a+3s-2)w_{2s-2} - (s-1)(a+s-1)(a+s-2)(2a+3s-4)w_{2s-4} - (s-1)(s-2)(a+s-1)(a+s-2)w_{2s-5} + (s-1)(s-2)(a+s-1)(a+s-2)^2(a+s-3)w_{2s-6},$$

$$s = 2, 3, \ldots,$$

with

$$k_0^* = 0,\ k_1^* = k_2^* = 1,\ k_0 = 1,\ k_1 = z^2,\ k_2 = z^2 + a(a+1);$$

$$k_s^* = 0,\ k_s = 0 \text{ for } s < 0.$$

The sequence $\{t_{2s}\}$ is increasing and $\{t_{2s-1}\}$ is convergent. In particular the coefficients in the recurrence relations for $\Phi(1, 1)$ are set out in Table 4, and they are to be read off from the penultimate row upwards. Thus, suppose we have found the values of k_s for $s = 1, 2, 3$ and 4; then from the column for $s = 5$ we see that

$k_5 = 1.k_4 + 14k_3 + 0.k_2 - 20.k_1 - 4k_0 + 0.k_{-1}$, and similarly for k_5^* .

TABLE 4.

RECURRENCE COEFFICIENTS FOR $\Phi(1, 1)$.

				0	24	72	432	864	2880	4800	12000	18000
			0	−4	−12	−36	−72	−144	−240	−400	−600	−900
	0	0	−8	−20	−84	−144	−360	−528	−1040	−1400	−2400	−3060
	0	0	0	0	0	0	0	0	0	0	0	0
	2	4	10	14	24	30	44	52	70	80	102	114
	1	1	1	1	1	1	1	1	1	1	1	1
s_6^7	2	3	4	5	6	7	8	9	10	11	12	13

A sequence of decreasing upper bounds is made available from

$$z^2\Phi(1, z) = 1 - 2\Phi(3, z)$$

using the appropriate second order C.F. arising from

$$\Phi(3, z) = \frac{1}{z} + \frac{3}{1} + \frac{1}{z} + \frac{4}{1} + \frac{2}{z} + \frac{5}{1} + \frac{3}{z} + \frac{6}{1} + \frac{4}{z} + \frac{7}{1} + \ldots .$$

The corresponding multipliers in the recurrence formulae are, when $z_1 = i$, $z_2 = -i$, those in Table 5.

TABLE 5.

RECURRENCE COEFFICIENTS FOR $\Phi(3, 1)$.

					480	240	360	2160	15120	10080	47040	33600	120960
				−24	−40	−120	−180	−360	−504	−840	−1120	−1680	−2160
			−96	−56	−440	−300	−1260	−936	−2856	−2240	−5600	−4560	−9936
			0	0	0	0	0	0	0	0	0	0	0
	12	8	28	22	50	42	78	68	112	100	152	138	198
	1	1	1	1	1	1	1	1	1	1	1	1	1
s	2	3	4	5	6	7	8	9	10	11	12	13	14

Proceeding in this way we are led to the approximations given in Table 6, which also includes similar ones for $\Phi(1, 2)$.

TABLE 6.

UPPER AND LOWER BOUNDS FOR $\Phi(1, 1)$ and $\Phi(1, 2)$.

	$\Phi(1, 1)$			$\Phi(1, 2)$	
s	(1)	(2)	s	(1)	(2)
4	0·52	0·74	4	0·196	0·205
8	0·612	0·653	8	0·1992	0·1998
12	0·6199	0·6284	12	0·19931	0·19954
16	0·6200	0·6227	16	0·19946	0·19953
20	0·6204	0·6217	20	0·19950	0·19952
24	0·6209	0·6216			

(1) and (2) refer to lower and upper bounds respectively.

It will be noticed that the rate of convergence for $\Phi(1, 1)$ is rather slow, and that after 24 terms we can only assert that $0{\cdot}6209 < \Phi(1,1) < 0{\cdot}6216$. For $\Phi(1, 2)$ the situation is better and twenty terms give accuracy in the fourth decimal place. Of course we could determine another set of upper bounds using (10), merely adding $s!\ s!/\{z^s k_{2s}\}$ to t_{2s}: however, there seems to be little improvement introduced in this way. According to Ser (1938) the values of the integrals are $\Phi(1, 1) = 0{\cdot}62145$, $\Phi(1, 2) = 0{\cdot}199510$.

7. *Conclusion.*

We intend to develop on another occasion the expansion of *generalised* C.F.'s using the compound determinants of § 2, each element of these determinants being a *recurrent*. We shall give two

types of recurrence formulae, and show that the evaluation of the convergents of a *generalised* C.F. of any order can be made a practical proposition.

I would like to put on record my appreciation of some stimulating remarks, and criticisms, of a referee.

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